Markov processes on vermiculated spaces

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Abstract. A general technique is given for constructing new Markov processes from existing ones. The new process and its state space are both projective limits of sequences built by an iterative scheme. The space at each stage in the scheme is obtained by taking disjoint copies of the space at the previous stage and quotienting to identify certain distinguished points. Away from the distinguished points, the process at each stage evolves like the one constructed at the previous stage on some copy of the previous state space, but when the process hits a distinguished point it enters at random another of the copies "pinned" at that point. Special cases of this construction produce diffusions on fractal-like objects that have been studied recently.

1. Introduction

In this paper we present a procedure for constructing new Markov processes from existing ones. This procedure can, for example, produce processes on rather exotic fractal-like spaces starting with processes on more familiar spaces such as Euclidean space.

The state space of the new process is built as a projective limit of an iterative scheme. The space at each stage of the iterative construction is produced by taking a disjoint collection of copies of the space coming from the previous stage and performing a quotient operation that identifies certain distinguished points.

A particular case of the state space construction, beginning with the real line in the iterative scheme, is presented in [Laa00] to produce examples of fractals with interesting analytic properties. A construction of graphs and random walks on them that is in a similar spirit to the continuous one given here is developed in [Bar01] to show that the only constraints on the volume growth and the anomalous diffusion exponent are those already known in the literature.

To give the flavour of the construction, we begin with a very simple example described in somewhat fanciful terms. Consider a "universe" that is a subinterval F of $\mathbb{R}$ and another "parallel" universe that is just a copy of F. At the same locations in each universe there is a set of "anchor points" B_1. Figure 1 shows a universe with 3 such anchor points.

In the language of innumerable science fiction stories, imagine that the two universes are connected by "wormholes" at the anchor points. This produces a

Figure 1: The space F with B_1 marked as *'s

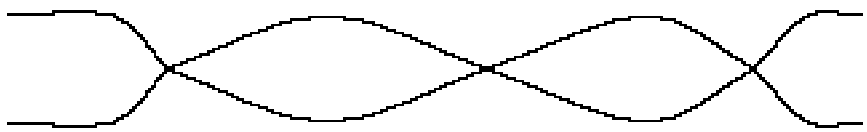


Figure 2: The space F_1 embedded in the plane

composite structure F_1 that, mathematically, is the product space $F \times \{0,1\}$ with points of the form $(b,0)$ and $(b,1)$, $b \in B_1$, identified. The space F_1 with the usual quotient topology looks like the subset of the plane drawn in Figure 2. For future purposes, it will be more convenient to represent F_1 schematically as in Figure 3 as two copies of F with the points at either end of an arrow identified.

At an intuitive level, there is an obvious way to "lift" a base Markov process on F to a Markov process on F_1. Namely, the lifted process evolves as the base process away from the anchor points, but when it hits an anchor point it chooses at random to either keep evolving in its current universe or to jump through the attached wormhole into the alternate universe. Of course, if the base process is something like Brownian motion, then there is some work that needs to be done to make this idea precise because of the fact that a Brownian motion returns to its starting point infinitely often in any neighbourhood of the origin. For Brownian motion, the technicalities involved in making sense of idea are of the same sort as those encountered in the construction of Walsh's spider and Brownian motion on more general graphs (see [Wal78, Var85, BPY89, DJ93, FW93, FW94, Kre95, Tsi97, BEK$^+$98, Eva00]). A substantial generalisation of the spider construction that applies to base processes on state spaces more general than the real line is given in [ES01]. That generalisation is basic to this paper and is reviewed in Section 2.

The construction that produced F_1 from F can be iterated. Suppose that F now has "first order" anchor points B_1 and "second order" anchor points B_2 as in Figure 4. An obvious quotient construction on $F_1 \times \{0,1\}$ produces a space F_2 shown schematically in Figure 5 as four copies of F with the points at either end of an arrow identified. The Markov process that was constructed on F_1 can

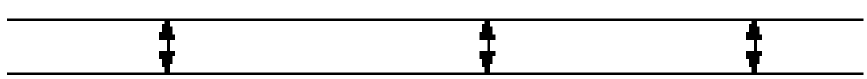


Figure 3: The space F_1 as two copies of F with identified points joined by $\updownarrow$

Figure 4: The space F with B_1 as $*$'s and B_2 as $+$'s

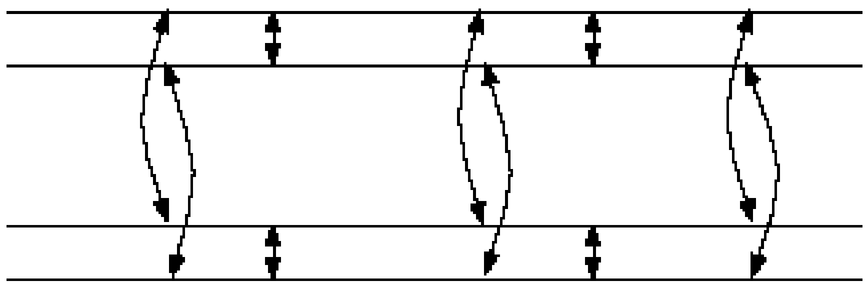


Figure 5: The space F_2 as four copies of F with identified points joined by $\updownarrow$

be lifted to F_2 by once again making suitable random choices whenever a second order anchor point is encountered.

Continuing in this manner produces a sequence of spaces $F_1, F_2, \ldots$ These spaces form a projective system and hence converge to a projective limit space F_∞. We will refer to the limit spaces produced by a generalisation of this construction (see Section 3) as *vermiculated* (that is, riddled with worm holes). Furthermore, the associated Markov processes have a natural projective structure, and consequently they give rise to a limit process on F_∞. The details are carried out in great generality in Section 4.

Properties of the limit Markov process for the case of a Brownian or Lévy base process will be investigated in a subsequent paper. The potential theory of such processes is particularly interesting. For example, it is possible that the base process hits points whereas the limit process does not if the anchor points and the number of copies at each stage in the iterative scheme are chosen correctly.

2. The pinching and twisting construction

In this section we review quickly a construction from [ES01] that produces one Markov process from another by means of a partial collapse of the state space and the introduction of appropriate extra randomisation. As we noted in the Introduction, this construction can be seen as a generalisation of the construction that produces Walsh's spider from Brownian motion on the line.

We begin with some topological ingredients. Let E and $\hat{E}$ be two Hausdorff, locally compact, second countable topological spaces. Thus E and $\hat{E}$ are, in particular, Polish (that is, metrisable as complete, separable metric spaces). Fix a continuous surjection $\psi : E \to \hat{E}$ such that $\psi^{-1}(K)$ is compact for any compact subset K of $\hat{E}$ and a closed set $A \subseteq E$ such that $\psi^{-1}(\psi(A)) = A$.

Set

$$\tilde{E} := (E \setminus A) \cup \psi(A),$$

this being a disjoint union. Define the map $\pi : E \to \tilde{E}$ by

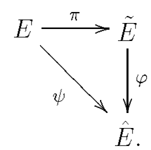
$$\pi(x) := \begin{cases} x, & \text{if } x \in E \setminus A, \\ \psi(x), & \text{if } x \in A, \end{cases}$$

and give $\tilde{E}$ the topology induced by π. That is, $U \subset \tilde{E}$ is open in the topology of $\tilde{E}$ if and only if $\pi^{-1}(U)$ is open in the topology of E. (Equivalenty, we can think of $\tilde{E}$ as the quotient topological space of the topological space E under the equivalence relation that declares two points x' and x'' equivalent if and only if $\pi(x') = \pi(x'')$.) Assume that $\tilde{E}$ with this topology is Hausdorff, locally compact, and second countable (and hence Polish).

Define a continuous map $\varphi : \tilde{E} \to \hat{E}$ by

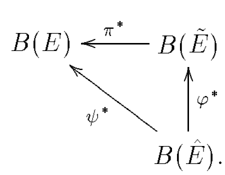
$$\varphi(x) := \begin{cases} \psi(x), & \text{if } x \in E \setminus A, \\ x, & \text{if } x \in \psi(A). \end{cases}$$

Then, $\psi = \varphi \circ \pi$, or, equivalently, we have the commutative diagram



Notation 2.1. Given a Hausdorff, locally compact, second countable topological space S, write $B(S)$ be the Banach space of bounded real-valued functions on S and let $B^+(S)$ be the collection of nonnegative elements of $B(S)$. Let $C_0(S)$ be the Banach space of real-valued continuous functions on S that vanish at infinity (if S is compact, then of course $C_0(S) = C(S)$, the Banach space of continuous functions on S). For any subset R of S, define $B(S;R) := \{f \in B(S) : f\big|_R \equiv 0\}$ and $C_0(S;R) := C_0(S) \cap B(S;R)$. If S' is a second locally compact space and ξ is a measurable map from S to S', we define $\xi^* : B(S') \to B(S)$ as $\xi^* f := f \circ \xi$. If ξ is continuous and $\xi^{-1}(K)$ is a compact subset of S for all compact subsets K of S', then $\xi^* : C_0(S') \to C_0(S)$.

Thus,



Moreover, $\varphi^* : C_0(\hat{E}) \to C_0(\tilde{E})$, $\pi^* : C_0(\tilde{E}) \to C_0(E)$, and $\psi^* : C_0(\hat{E}) \to C_0(E)$. Define $\breve{\pi}_* : B(E; A) \to B(\tilde{E}; \psi(A))$ by

$$(\breve{\pi}_* f)(x) := \begin{cases} f(x), & \text{if } x \in E \setminus A, \\ 0, & \text{if } x \in \psi(A). \end{cases}$$

Note that $\breve{\pi}_* : C_0(E; A) \to C_0(\tilde{E}; \psi(A))$

We now introduce the probabilistic ingredients of the construction. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ (resp. $\hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\mathbb{P}}^x)$) be a quasi-left-continuous Borel right process with state space E (resp. $\hat{E}$) and resolvent $(U^\alpha)_{\alpha>0}$ (resp. $(\hat{U}^\alpha)_{\alpha>0}$). Write $(V^\alpha)_{\alpha>0}$ (resp. $(\hat{V}^\alpha)_{\alpha>0}$) for the the resolvent of X stopped on hitting A (resp. for $\hat{X}$ stopped on hitting $\psi(A)$). Assume that $V^\alpha C_0(E) \subseteq C_0(E)$ and $\hat{U}^\alpha C_0(\hat{E}) \subseteq C_0(\hat{E})$ for all $\alpha > 0$.

Let $k : \hat{E} \times \mathcal{B}(E) \to \mathbb{R}$ be a probability kernel. Define a linear operator $K : B(E) \to B(\hat{E})$ by $Kf(x) := \int_{y \in E} f(y) k(x, dy)$. The appropriate diagram is thus

$$B(E) \xrightarrow{\;K\;} B(\hat{E}).$$

We assume that

$$k(x, \psi^{-1}\{x\}) = 1$$

for all $x \in \hat{E}$ and that $KC_0(\hat{E}) \subseteq C_0(E)$.

Assume that the resolvents (U^α), $(\hat{U}^\alpha)$ and (V^α), $(\hat{V}^\alpha)$ and the kernel K satisfy the *Dynkin intertwining relation*

$$U^\alpha \psi^* = \psi^* \hat{U}^\alpha.$$

(which implies $V^\alpha \psi^* = \psi^* \hat{V}^\alpha$) and the *Carmona–Petit–Yor intertwining relation*

$$KU^\alpha = \hat{U}^\alpha K \quad \text{and} \quad KV^\alpha = \hat{V}^\alpha K.$$

The following result is proved in [ES01]. Intuitively, it describes the construction of a process on $\tilde{E}$ that evolves as X on $E \setminus A$ and as $\hat{X} \equiv \psi \circ X$ on $\psi(A)$. When this process passes from $\psi(A)$ into $E \setminus A$ it undergoes a random twist according to the kernel k. We refer the reader to [ES01] for more details and several examples.

Theorem 2.2. *Under the above assumptions, the following hold.*

a) There is a quasi–left–continuous Borel right process $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{\mathbb{P}}^x)$ *with resolvent* $(\tilde{U}^\alpha)_{\alpha>0}$ *given by*

$$\tilde{U}^\alpha = \breve{\pi}_* V^\alpha \pi^* \left(I_{\tilde{E}} - \varphi^* K \pi^*\right) + \varphi^* \hat{U}^\alpha K \pi^*.$$

Moreover, $\tilde{U}^\alpha C_0(\tilde{E}) \subseteq C_0(\tilde{E})$.

b) For each $x \in \tilde{E}$, the law of $\varphi \circ \tilde{X}$ under $\tilde{\mathbb{P}}^x$ coincides with that of $\hat{X}$ under $\hat{\mathbb{P}}^{\varphi(x)}$.

c) Define a stopping times by

$$T := \inf\{t \geq 0 : X_t \in A\},$$
$$\tilde{T} := \inf\left\{t \geq 0 : \tilde{X}_t \in \psi(A)\right\}.$$

For each $x \in E$, the law of $\{\tilde{X}_t; 0 \leq t < \tilde{T}\}$ under $\tilde{\mathbb{P}}^{\pi(x)}$ is equal to the law of $\{\pi(X_t); 0 \leq t < T\}$ under $\mathbb{P}^x$.

The following stopped version of Theorem 2.2 is not proved in [ES01], but follows using the same ideas.

Corollary 2.3. *Suppose that the above assumptions hold. Let $B \subseteq E$ be a closed set such that $\psi^{-1}(\psi(B)) = B$. Write $(\underline{V}^\alpha)_{\alpha>0}$ for the resolvent of X stopped on hitting $A \cup B$. Denote by $(\underline{\tilde{U}}^\alpha)_{\alpha>0}$ (resp. $(\underline{\hat{U}}^\alpha)_{\alpha>0}$) for the resolvent of $\tilde{X}$ (resp. $\hat{X}$) stopped on hitting $\pi(B)$ (resp. $\psi(B)$). Suppose that $\underline{V}^\alpha C_0(E) \subseteq C_0(E)$ and $\underline{\hat{U}}^\alpha C_0(\hat{E}) \subseteq C_0(\hat{E})$ for all $\alpha > 0$. Then*

$$\underline{\tilde{U}}^\alpha = \check{\pi}_* \underline{V}^\alpha \pi^* \left(I_{\tilde{E}} - \varphi^* K \pi^*\right) + \varphi^* \underline{\hat{U}}^\alpha K \pi^*,$$

and $\underline{\tilde{U}}^\alpha C_0(\tilde{E}) \subseteq C_0(\tilde{E})$ for all $\alpha > 0$.

3. Vermiculated spaces

We begin with a generalisation of the iterative state space construction outlined in the Introduction.

Let F be a Hausdorff, locally compact, second countable topological space. Suppose that $B_1, B_2, \ldots$ are closed subsets of F. Let $G_1, G_2, \ldots$ be Hausdorff, compact, second countable topological spaces. The space G_n will index the collection of "alternate universes" at stage n of the construction. In the example described in the Introduction, $G_1 = G_2 = \{0, 1\}$.

Put $F_0 := F$ and

$$E_1 := F_0 \times G_1$$
$$\hat{E}_1 := F_0,$$

and

$$A_1 := B_1 \times G_1 \subseteq E_1.$$

Define $\psi_1 : E_1 \to \hat{E}_1$ by

$$\psi_1(y, z) = y.$$

Now apply the general state space construction of Section 2 with the ingredients $E = E_1$, $\hat{E} = \hat{E}_1$, $A = A_1$, and $\psi = \psi_1$. It is clear that the conditions of the construction hold.

Write $\tilde{E}_1$ for the resulting space denoted by $\tilde{E}$ in the general construction, and π_1, φ_1 for the maps denoted by π, φ in the general construction. Thus

$$\tilde{E}_1 = (E_1 \setminus A_1) \cup \psi_1(A_1) = ((F_0 \setminus B_1) \times G_1) \cup B_1$$

and $\pi_1 : E_1 \to \tilde{E}_1$ is given by

$$\pi_1(y, z) = \begin{cases} (y, z), & \text{if } y \in F_0 \setminus B_1, \\ y, & \text{if } y \in B_1; \end{cases}$$

and $\tilde{E}_1$ is equipped with the topology induced by π_1. Set $F_1 := \tilde{E}_1$ and write $\varphi_1 : F_1 \to F_0$ for the map denoted by φ in the general construction. Thus

$$\varphi_1(y, z) = y, \quad (y, z) \in E_1 \setminus A_1 = (F_0 \setminus B_1) \times G_1,$$
$$\varphi_1(y) = y, \quad y \in \psi_1(A_1) = B_1.$$

Suppose now that Hausdorff, locally compact, second countable topological spaces F_m, $0 \le m \le n$, and continuous surjections $\varphi_m : F_m \to F_{m-1}$, $1 \le m \le n$, have already been constructed. Define F_{n+1} and a continuous surjection $\varphi_{n+1} : F_{n+1} \to F_n$ as follows.

Put

$$E_{n+1} := F_n \times G_{n+1},$$
$$\hat{E}_{n+1} := F_n,$$

and

$$A_{n+1} := \varphi_{n,0}^{-1}(B_{n+1}) \times G_{n+1} \subseteq E_{n+1},$$

where

$$\varphi_{j,i} := \varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_j, \quad 0 \le i < j \le n.$$

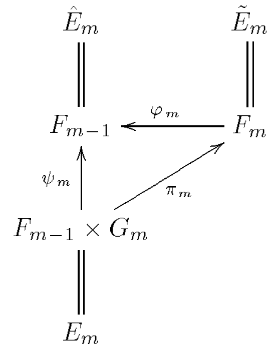
Define $\psi_{n+1} : E_{n+1} \to \hat{E}_{n+1}$ by

$$\psi_{n+1}(y, z) = y.$$

Now apply the general state space construction of Section 2 with the ingredients $E = E_{n+1}$, $\hat{E} = \hat{E}_{n+1}$, $A = A_{n+1}$, and $\psi = \psi_{n+1}$. Once again, it is clear that the conditions of the construction hold. Set $F_{n+1} := \tilde{E}_{n+1}$ and write φ_{n+1} for the map denoted by φ in the general construction; that is,

$$\varphi_{n+1}(y, z) = y, \quad (y, z) \in E_{n+1} \setminus A_{n+1} = (F_n \setminus \varphi_{n,0}^{-1}(B_{n+1})) \times G_{n+1},$$
$$\varphi_{n+1}(y) = y, \quad y \in \psi_{n+1}(A_{n+1}) = \varphi_{n,0}^{-1}(B_{n+1}).$$

The salient points of the construction are summarised as follows:



The sequence of spaces $(F_n)_{n=0}^{\infty}$ equipped with the maps $(\varphi_{j,i})_{0 \le i < j < \infty}$ is a *projective system* of topological spaces (sometimes also called an *inverse system*). Therefore this system has a *projective limit* topological space $F_\infty := \lim_{\leftarrow} F_n$ (also called an *inverse limit*) equipped with a family of continuous surjections $\Phi_n : F_\infty \to F_n$, $0 \le n \le \infty$ satisfying $\varphi_{j,i} \circ \Phi_j = \Phi_i$, $0 \le i < j < \infty$. By general facts about projective limits, the space F_∞ is Hausdorff, locally compact and second countable (see Section 2-14 of [HY61]).

4. Construction of a projective limit process

We continue with the development begun in Section 3.

Let ξ be a quasi-left-continuous Borel right process with state-space F. The process ξ is the base process that we will successively lift up to $F_1, F_2, \ldots$ in the manner outlined in the Introduction. Let μ_n be a Borel probability measure on G_n, $n \ge 1$. Recalling that G_n indexes the various alternate universes at the n^{th} stage of the iterative part of the state space construction, the probability measure μ_n describes how an alternate universe is chosen when the n^{th} stage process hits an anchor point. In the example described in the Introduction, μ_1 and μ_2 are both the uniform measure on $\{0, 1\}$.

Assumption 4.1. Write $\mathcal{C}$ for the collection consisting of the empty set and finite unions of sets drawn from $B_1, B_2, \ldots$ Assume for each $C \in \mathcal{C}$ that the resolvent of ξ stopped on hitting C maps $C_0(E)$ into itself.

Example 4.2. Assumption 4.1 holds for $F = \mathbb{R}$, ξ a standard Brownian motion, and any closed sets $B_1, B_2, \ldots \subseteq \mathbb{R}$.

Set X^0 to be the Markov process ξ on $F_0 = F$. Suppose that quasi-left-continuous Borel right processes $X^m = (\Omega^m, \mathcal{F}^m, \mathcal{F}_t^m, X_t^m, \theta_t^m \mathbb{P}_m^x)$, $0 \le m \le n$, have been defined. For $C \in \mathcal{C}$, write $(U_{C,m}^\alpha)_{\alpha > 0}$ for the resolvent of X^m stopped on hitting $\varphi_{m,0}^{-1}(C)$. Suppose further that $U_{C,m}^{\alpha} C_0(F_m) \subseteq C_0(F_m)$ for all $C \in \mathcal{C}$, $\alpha > 0$, $0 \le m \le n$.

Consider the construction of Section 2 with the ingredients:

- $E = F_n \times G_{n+1}$,
- $\hat{E} = F_n$,
- $A = \varphi_{n,0}^{-1}(B_{n+1}) \times G_{n+1} \subseteq E$,
- $\psi(y, z) = y$, $(y, z) \in F_n \times G_{n+1}$,
- X under $\mathbb{P}^{(y,z)}$ has the law of $\{(X^n, z) : t \ge 0\}$ when X_n is under $\mathbb{P}_n^y$,
- $\hat{X} = X^n$,
- $k(y, \cdot) = \delta_y \otimes \mu_{n+1}$.

It is clear that the conditions of Theorem 2.2 hold. Moreover, the conditions of Corollary 2.3 hold with the set B given by $\varphi_{n,0}^{-1}(C) \times G_{n+1}$ for any $C \in \mathcal{C}$.

Let $X_{n+1} = (\Omega^{n+1}, \mathcal{F}^{n+1}, \mathcal{F}_t^{n+1}, X_t^{n+1}, \theta_t^{n+1}, \mathbb{P}_{n+1}^x)$ be the quasi-left-continuous Borel right processes on $\tilde{E} = F_{n+1}$ produced by Theorem 2.2. Writing $(U_{C,n+1}^{\alpha})_{\alpha > 0}$ for the resolvent of X^{n+1} stopped on hitting $\varphi_{n+1,0}^{-1}(C)$, $C \in \mathcal{C}$, Corollary 2.3 guarantees that $U_{C,n+1}^{\alpha} C_0(F_{n+1}) \subseteq C_0(F_{n+1})$ for all $\alpha > 0$.

Denoting by $(U_n^{\alpha})_{\alpha > 0}$ the resolvent of X^n, it follows from Theorem 2.2 that

$$U_j^{\alpha} \varphi_{j,i}^* = \varphi_{j,i}^* U_i^{\alpha}, \quad 0 \le i < j < \infty. \tag{4.1}$$

The subspace $\bigcup_n \Phi_n^* C_0(F_n)$ is dense in $C_0(F_\infty)$ by construction. By (4.1), there are well-defined linear operators U_∞^{α}, $\alpha > 0$, defined on $\bigcup_n \Phi_n^* C_0(F_n)$ such that

$$U_\infty^{\alpha} \Phi_n^* = \Phi_n^* U_n^{\alpha}, \text{ for all } n. \tag{4.2}$$

It is clear that each U_∞^{α} extends by continuity to a Markov operator on $C_0(F_\infty)$, and this extension still satisfies the Dynkin intertwining relation (4.2). By (4.2) and the resolvent equation for $(U_n^{\alpha})_{\alpha > 0}$,

$$U_\infty^{\alpha} \Phi_n^* = \Phi_n^* U_n^{\alpha} = \Phi_n^* \left(U_n^{\beta} + (\beta - \alpha) U_n^{\alpha} U_n^{\beta} \right) = \left(U_\infty^{\beta} + (\beta - \alpha) U_\infty^{\alpha} U_\infty^{\beta} \right) \Phi_n^*$$

and hence, by continuity, $(U_\infty^{\alpha})_{\alpha > 0}$ obeys the resolvent equation on $C_0(F_\infty)$. Again by continuity,

$$\lim_{\alpha \to \infty} U_\infty^{\alpha} f = f \text{ pointwise, } f \in C_0(F_\infty). \tag{4.3}$$

Standard arguments (see Theorem 9.26 of [Sha88]) and (4.2) now give the following.

Theorem 4.3. *Under Assumption 4.1 there is a quasi-left-continuous Borel right process*

$$X^\infty = (\Omega^\infty, \mathcal{F}^\infty, \mathcal{F}_t^\infty, X_t^\infty, \theta_t^\infty, \mathbb{P}_\infty^x)$$

on F_∞ *with resolvent* $(U_\infty^\alpha)_{\alpha>0}$. *The law of* $\Phi_n \circ X^\infty$ *under* $\mathbb{P}_\infty^x$ *is that of* X^n *under* $\mathbb{P}_n^{\Phi_n(x)}$.

Remark 4.4. It is worth pointing out what can go wrong if Assumption 4.1 does not hold. For example, consider the first stage of the inductive part of the construction with $F = \mathbb{R}^2$, $B_1 = \{0\}$, $G_1 = \{0,1\}$, $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$, and ξ a standard planar Brownian motion. If our construction "worked" in this instance it would produce a process X^1 that, when started at the anchor point 0, picks at random between the two copies of $\mathbb{R}^2$ pinned at 0 and then never leaves this copy. The identity of the chosen copy of $\mathbb{R}^2$ is thus a non-trivial random variable measurable with respect to the germ σ-field of X^1 at time 0, and so even the Blumenthal zero–one law fails for X^1.

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References

[Bar01] M.T. Barlow. Which values of the volume growth and escape time exponent are possible for a graph? Preprint, 2001.

[BEK+98] M.T. Barlow, M. Émery, F.B. Knight, S. Song, and M. Yor. Autour d'un théorème de Tsirelson sur les filtrations Browniennes et non Browniennes. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Mathematics*, pages 264–305. Springer, Berlin, 1998.

[BPY89] M.T. Barlow, J. Pitman, and M. Yor. On Walsh's Brownian motions. In *Séminaire de Probabilités, XXXII*, volume 1372 of *Lecture Notes in Mathematics*, pages 275–293. Springer, Berlin, New York, 1989.

[DJ93] D.S. Dean and K.M. Jansons. Brownian excursions on combs. *J. Statist. Phys.*, 70:1313–1332, 1993.

[ES01] S.N. Evans and R.B. Sowers. Pinching and twisting Markov processes. Technical Report 587, Department of Statistics, U.C. Berkeley, 2001. `www.stat.berkeley.edu/users/evans/587.ps.Z`.

[Eva00] Steven N. Evans. Snakes and spiders: Brownian motion on $\mathbb{R}$-trees. *Probab. Theory Related Fields*, 117(3):361–386, 2000.

[FW93] M.I. Freidlin and A.D. Wentzell. Diffusion processes on graphs and the averaging principle. *Ann. Probab.*, 21:2215–2245, 1993.

[FW94] Mark I. Freidlin and Alexander D. Wentzell. Random perturbations of Hamiltonian systems. *Mem. Amer. Math. Soc.*, 109(523):viii+82, 1994.

[HY61] J.G. Hocking and G.S. Young. *Topology.* Addison–Wesley, Reading, MA, 1961.

[Kre95] W.B. Krebs. Brownian motion on the continuum tree. *Probab. Theory Related Fields*, 101:421–433, 1995.

[Laa00] T. J. Laakso. Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[Sha88] M. Sharpe. *General Theory of Markov Processes*. Academic Press, San Diego, 1988.

[Tsi97] B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.*, 7:1096–1142, 1997.

[Var85] Nicholas Th. Varopoulos. Long range estimates for Markov chains. *Bull. Sci. Math. (2)*, 109(3):225–252, 1985.

[Wal78] J.B. Walsh. A diffusion with discontinuous local time. In *Temps Locaux, Astérisque*, volume 52–53. Société Mathématique de France, Paris, 1978.

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