Bivariate Uniqueness in the Logistic Recursive Distributional Equation

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Abstract

In this work we prove the *bivariate uniqueness* property of the Logistic fixed-point equation, which arise in the study of the *random assignment problem*, as discussed by Aldous [4]. Using this and the general framework of Aldous and Bandyopadhyay [2], we then conclude that the associated *recursive tree process* is *endogenous*, and hence the Logistic variables defined in Aldous' work [4] are measurable with respect to the σ-field generated by the edge weights. The method involves construction of an explicit recursion to show the uniqueness of the associated integral equation.

Key words and phrases. Bivariate uniqueness, distributional identity, fixed-point equation, Logistic distribution, measurability, non-linear integral equation, Poisson weighted infinite tree, random assignment problem.

1 Introduction

Our result is much easier to state than to motivate, so we will first state the main result (Theorem 1) of this work.

Theorem 1 *Let $0 < \xi_1 < \xi_2 < \cdots$ be points of a Poisson point process of rate 1 on $(0, \infty)$. Let $(X, Y), ((X_j, Y_j))_{j \geq 1}$ be independent random variables with some common distribution ν on $\mathbb{R}^2$, which are independent of $(\xi_j)_{j \geq 1}$. Then*

$$\left(\begin{array}{c} X \\ Y \end{array} \right) \stackrel{d}{=} \left(\begin{array}{c} \min\limits_{j \geq 1} (\xi_j - X_j) \\ \min\limits_{j \geq 1} (\xi_j - Y_j) \end{array} \right), \tag{1}$$

if and only if $\nu = \mu^{\nearrow}$, where $\mu^{\nearrow}$ is defined as the joint distribution of (Z, Z) on $\mathbb{R}^2$, with $Z \sim$ Logistic distribution, that is, the distribution function of Z is given by

$$H(x) := \mathbf{P}(Z \leq x) = \frac{1}{1 + e^{-x}}, \; x \in \mathbb{R} \tag{2}$$

Theorem 1 is a concrete result falling within the framework of *recursive distributional equations* surveyed in [2]. This particular problem arose in the study of a classical problem of combinatorial optimization, namely, the *mean-field random assignment problem.* The following section develops background and provides the details of our motivation for Theorem 1. In Section 3 we give a proof of Theorem 1 using analytic techniques. Some technical results which are not terribly important, but are needed for the proof, are given separately in Section 4.

2 Background and Motivation

This section mainly provides background and motivation for Theorem 1.

For a given $n \times n$ matrix of costs (C_{ij}), consider the problem of assigning n jobs to n machines in the most "cost effective" way. Thus the task is to find a permutation π of $\{1, 2, \ldots, n\}$, which solves the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^{n} C_{i, \pi(i)}. \tag{3}$$

This problem has been extensively studied in literature for a fixed cost matrix, and there are various algorithms to find the optimal permutation π. A probabilistic model for the assignment problem can be obtained by assuming that the costs are independent random variables each with distribution

Uniform$[0,1]$. Although this model appears to be rather simple, careful investigations of it in the last few decades have proven that, it has enormous richness in its structure. For a careful survey and other related works see [8, 3].

Our interest in this problem is from another perspective. In 2001 Aldous [4] showed that

$$\lim_{n \to \infty} \mathbf{E}[A_n] = \zeta(2) = \frac{\pi^2}{6}, \tag{4}$$

confirming the earlier work of Mézard and Parisi [6], where they computed the same limit using some non-rigorous arguments based on the *replica method* [7]. In an earlier work Aldous [1] showed that the limit of $\mathbf{E}[A_n]$ as $n \to \infty$ exists for any cost distribution, and does not depend on the specifics of it, except only on the value of its density at 0, provided it exists and is strictly positive. So for calculation of the limiting constant one can assume that C_{ij}'s are independent and each has Exponential distribution with mean n, and re-write the objective function A_n in the normalized form,

$$A_n = \min_{\pi} \frac{1}{n} \sum_{i=1}^{n} C_{i,\pi(i)}. \tag{5}$$

Aldous [4] identified the limit constant $\zeta(2)$ in terms of an optimal matching problem on a limit infinite tree with random edge weights. This structure is called *Poisson Weighted Infinite Tree*, or, *PWIT*, it is described as follows (see the survey of Aldous and Steele [3] for a more friendly account).

> Let $\mathcal{T} := (\mathcal{V}, \mathcal{E})$ be the canonical infinite rooted tree with vertex set $\mathcal{V} := \cup_{m=0}^{\infty} \mathbb{N}^m$ (where $\mathbb{N}^0 := \{\emptyset\}$), and edge set $\mathcal{E} := \{e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N}\}$. We consider $\emptyset$ as the root of the tree, and will write $\emptyset j = j \ \forall j \in \mathbb{N}$. For every vertex $\mathbf{i} \in \mathcal{V}$, let $(\xi_{\mathbf{i}j})_{j \geq 1}$ be points of independent Poisson process of rate 1 on $(0, \infty)$. Define the weight of the edge $e = (\mathbf{i}, \mathbf{i}j)$ as $\xi_{\mathbf{i}j}$.

Aldous [4] showed that on a PWIT one can construct random variables $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ taking values in $\mathbb{R}$, such that

- $X_{\mathbf{i}} = \min_{j \geq 1} \left(\xi_{\mathbf{i}j} - X_{\mathbf{i}j} \right), \ \ \forall \ \mathbf{i} \in \mathcal{V}.$
- $X_{\mathbf{i}}$ is independent of $\left\{ \left(\xi_{\mathbf{i}'j} \right)_{j \geq 1} \,\middle|\, |\mathbf{i}'| < |\mathbf{i}| \right\}$, for all $\mathbf{i} \in \mathcal{V} \backslash \{\emptyset\}$; where we write $|\mathbf{i}| = d$ if $\mathbf{i} \in \mathbb{N}^d$.
- $X_{\mathbf{i}} \sim$ Logistic distribution.

As defined in Theorem 1, we say a random variable taking real values has Logistic distribution if it has distribution function given by (2).

The key "calculation" ingredient of the construction above is the following *recursive distributional equation (RDE)*, which we will refer as the *Logistic RDE*.

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right), \tag{6}$$

where $\left(\xi_j\right)_{j \geq 1}$ are points of a Poisson point process of rate 1 on $(0, \infty)$, and are independent of $\left(X_j\right)_{j \geq 1}$, which are independent and identically distributed with same law as of X. It is easy to show [4] that (6) has unique solution as Logistic distribution. The existence of $X_{\mathbf{i}}$'s then follows from Kolmogorov's consistency. The construction of the $(X_{\mathbf{i}})$ from the Logistic RDE is abstracted in [2]; to any RDE one can associate a *recursive tree process (RTP)*. In [4] Aldous has heuristic interpretation of $X_{\mathbf{i}}$'s through the edge weights. Thus it is natural to ask if they are actually measurable with respect to the sigma-field generated by the edge weights. We note that the abstract construction of the $X_{\mathbf{i}}$'s using Kolmogorov's consistency does not throw any light towards measurability issues. Notationally, if

$$\mathcal{G} := \sigma \left(\left\{ \left(\xi_{\mathbf{i}j}\right)_{j \geq 1} \,\middle|\, \mathbf{i} \in \mathcal{V} \right\} \right), \tag{7}$$

then the question above can be restated as

$$\text{is } X_{\emptyset} \text{ measurable with respect to the } \sigma\text{-field } \mathcal{G} \text{ ?} \tag{8}$$

Our work is motivated to answer this question of Aldous (see remarks (4.2.d) and (4.2.e) in [4]). Remarkably enough, one can show in the abstract setting of RTP that the question (8) has an affirmative answer if and only if the "bivariate uniqueness" property of Theorem 1 holds. This has been extensively discussed in the work of Aldous and Bandyopadhyay [2], whose Theorem 11 specializes to our current setting as follows.

Theorem 2 *Suppose $\mathfrak{S}$ is the set of all probabilities on $\mathbb{R}^2$ and we define $\Lambda : \mathfrak{S} \to \mathfrak{S}$ as*

$$\Lambda(\nu) \stackrel{d}{=} \left(\begin{array}{c} \min_{j \geq 1} \left(\xi_j - X_j \right) \\ \min_{j \geq 1} \left(\xi_j - Y_j \right) \end{array} \right), \tag{9}$$

where $\left(\xi_j\right)_{j \geq 1}$ are points of a Poisson process with mean intensity 1 *on $(0, \infty)$, and are independent of $\left(X_j, Y_j\right)_{j \geq 1}$, which are i.i.d with distribution ν on $\mathbb{R}^2$. Suppose further that Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ of $\mathfrak{S}$ defined as*

$$\mathfrak{S}^\star := \left\{ \nu \in \mathfrak{S} \,\middle|\, \textit{both the marginals of } \nu \textit{ are Logistic distribution} \right\}. \tag{10}$$

If the fixed-point equation $\Lambda(\nu) = \nu$ has unique solution as $\mu^{\nearrow}$ (as defined in Theorem 1) then $X_\emptyset$ as defined above is measurable with respect to the σ-field $\mathcal{G}$.

Notice that Theorem 1 basically states that $\Lambda(\nu) = \nu$ has unique solution as $\mu^{\nearrow}$. Further it is easy to see that the operator Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ (see Proposition 9 of Section 4 for a proof). Thus using Theorem 2 we get the following immediate corollary of Theorem 1 which answers the question (8) affirmatively, and hence proving that the RTP associated with Logistic RDE (6) is *endogenous*.

Corollary 3 *$X_\emptyset$ as defined above is measurable with respect to the σ-field $\mathcal{G}$, generated by the edge weights of the PWIT.*

3 Proof of Theorem 1

First observe that if the equation (1) has a solution then, the marginal distributions of X and Y solve the Logistic RDE, and hence they are both Logistic. Further by inspection $\mu^{\nearrow}$ is a solution of (1). So it is enough to prove that $\mu^{\nearrow}$ is the only solution of (1).

Let ν be a solution of (1). Notice that the points $\{(\xi_j; (X_j, Y_j)) \,|\, j \geq 1\}$ form a Poisson point process, say $\mathcal{P}$, on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $\rho(t; (x, y))\, dt\, d(x, y) := dt\, \nu(d(x, y))$. Thus if $G(x, y) := \mathbf{P}\,(X > x, Y > y)$, for $x, y \in \mathbb{R}$, then

$$\begin{aligned}
G(x,y) &= \mathbf{P}\left(\min_{j\geq 1}(\xi_j - X_j) > x, \text{ and, } \min_{j\geq 1}(\xi_j - Y_j) > y\right) \\
&= \mathbf{P}\left(\text{No points of } \mathcal{P} \text{ are in } \left\{(t;(u,v)) \,\Big|\, t-u \leq x, \text{ or, } t-v \leq y\right\}\right) \\
&= \exp\left(-\iiint_{t-u\leq x, \text{ or, } t-v\leq y} \rho(t;(u,v))\, dt\, d(u,v)\right) \\
&= \exp\left(-\int_0^\infty \left[\overline{H}(t-x) + \overline{H}(t-y) - G(t-x,t-y)\right]\, dt\right) \\
&= \overline{H}(x)\,\overline{H}(y)\, \exp\left(\int_0^\infty G(t-x,t-y)\, dt\right). \qquad (11)
\end{aligned}$$

The last equality follows from properties of the Logistic distribution (see Fact 1 of appendix). For notational convenience in this paper we will write $\overline{F}\,(\cdot) := 1 - F\,(\cdot)$, for any distribution function F.

The following simple observation reduces the bivariate problem to a univariate problem.

Lemma 4 *For any two random variables U and V, $U = V$ a.s. if and only if $U \overset{d}{=} V \overset{d}{=} U \wedge V$.*

Proof : First of all if $U = V$ a.s. then $U \wedge V = U$ a.s.

Conversely suppose that $U \overset{d}{=} V \overset{d}{=} U \wedge V$. Fix a rational q, then under our assumption,

$$\begin{aligned}
\mathbf{P}\left(U \leq q < V\right) &= \mathbf{P}\left(V > q\right) - \mathbf{P}\left(U > q, V > q\right) \\
&= \mathbf{P}\left(V > q\right) - \mathbf{P}\left(U \wedge V > q\right) \\
&= 0
\end{aligned}$$

A similar calculation will show that $\mathbf{P}\left(V \leq q < U\right) = 0$. These are true for any rational q, thus $\mathbf{P}\left(U \neq V\right) = 0$.

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Thus if we can show that $X \wedge Y$ also has Logistic distribution, then from the lemma above we will be able to conclude that $X = Y$ a.s., and hence the proof will be complete. Put $g(\cdot) := \mathbf{P}\left(X \wedge Y > \cdot\right)$, we will show $g = \overline{H}$. Now, for every fixed $x \in \mathbb{R}$, $g(x) = G(x, x)$ by definition. So using (11) we get

$$g(x) = \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} g(s)\, ds\right), \; x \in \mathbb{R}. \tag{12}$$

Notice that from (A1) (see Fact 3 of appendix) $g = \overline{H}$ is a solution of this non-linear integral equation (12), which corresponds to the solution $\nu = \mu^{\nearrow}$ of the original equation (1). To complete the proof of Theorem 1 we need to show that this is the only solution. For that we will prove that the operator associated with (12) (defined on an appropriate space) is monotone and has unique fixed-point as $\overline{H}$. The techniques we will use here are similar to Eulerian recursion, and are heavily based on analytic arguments.

Let $\mathfrak{F}$ be the set of all functions $f : \mathbb{R} \to [0, 1]$ such that

- $\overline{H}^2(x) \leq f(x) \leq \overline{H}(x), \; \forall\, x \in \mathbb{R}$,
- f is a tail of a distribution, that is, $\exists$ random variable say W such that $f(x) = \mathbf{P}\left(W > x\right), \, x \in \mathbb{R}$.

Observe that by definition $\overline{H} \in \mathfrak{F}$. Further, from (12) it follows that $g(x) \geq \overline{H}^2(x),\ \forall\, x \in \mathbb{R}$, as well as, $g(x) = \mathbf{P}\left(X \wedge Y > x\right) \leq \mathbf{P}\left(X > x\right) = \overline{H}(x),\ \forall\, x \in \mathbb{R}$. So it is appropriate to search for solutions of (12) in $\mathfrak{F}$.

Let $T : \mathfrak{F} \to \mathfrak{F}$ be defined as

$$T(f)(x) := \overline{H}^2(x)\, \exp\left(\int_{-x}^{\infty} f(s)\, ds\right),\ x \in \mathbb{R}. \tag{13}$$

Proposition 10 of Section 4 shows that T does indeed map $\mathfrak{F}$ into itself. Observe that the equation (12) is nothing but the fixed-point equation associated with the operator T, that is,

$$g = T(g) \ \text{ on } \ \mathfrak{F}. \tag{14}$$

We here note that using (A1) (see Fact 3 of appendix) T can also be written as

$$T(f)(x) := \overline{H}(x)\, \exp\left(-\int_{-x}^{\infty} \left(\overline{H}(s) - f(s)\right)\, ds\right),\ x \in \mathbb{R}, \tag{15}$$

which will be used in the subsequent discussion.

Define a partial order $\preccurlyeq$ on $\mathfrak{F}$ as, $f_1 \preccurlyeq f_2$ in $\mathfrak{F}$ if $f_1(x) \leq f_2(x),\ \forall\, x \in \mathbb{R}$, then the following result holds.

Lemma 5 *T is a monotone operator on the partially ordered set $(\mathfrak{F}, \preccurlyeq)$.*

Proof : Let $f_1 \preccurlyeq f_2$ be two elements of $\mathfrak{F}$, so from definition $f_1(x) \leq f_2(x),\ \forall\, x \in \mathbb{R}$. Hence

$$\begin{array}{rccccl} & \int_{-x}^{\infty} f_1(s)\, ds & \leq & \int_{-x}^{\infty} f_2(s)\, ds, & \forall\, x \in \mathbb{R} \\ \Rightarrow & T(f_1)(x) & \leq & T(f_2)(x), & \forall\, x \in \mathbb{R} \\ \Rightarrow & T(f_1) & \preccurlyeq & T(f_2). & \end{array}$$

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Put $f_0 = \overline{H}^2$, and for $n \in \mathbb{N}$, define $f_n \in \mathfrak{F}$ recursively as, $f_n = T(f_{n-1})$. Now from Lemma 5 we get that if g is a fixed-point of T in $\mathfrak{F}$ then,

$$f_n \preccurlyeq g,\ \forall\ n \geq 0. \tag{16}$$

If we can show $f_n \to \overline{H}$ pointwise, then using (16) we will get $\overline{H} \preccurlyeq g$, so from definition of $\mathfrak{F}$ it will follow that $g = \overline{H}$, and our proof will be complete. For that, the following lemma gives an explicit recursion for the functions $\{f_n\}_{n \geq 0}$.

Lemma 6 *Let $\beta_0(s) = 1 - s$, $0 \le s \le 1$. Define recursively*

$$\beta_n(s) := \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right) dw,\ 0 < s \le 1. \tag{17}$$

Then for $n \ge 1$,

$$f_n(x) = \overline{H}(x)\,\exp\left(-\beta_{n-1}(\overline{H}(x))\right),\ x \in \mathbb{R}. \tag{18}$$

Proof : We will prove this by induction on n. Fix $x \in \mathbb{R}$, for $n = 1$ we get

$$\begin{aligned}
f_1(x) &= T(f_0)(x) \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\left(\overline{H}(s) - \overline{H}^2(s)\right) ds\right) \qquad [\text{ using (15) }] \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\overline{H}(s)\,\left(1 - \overline{H}(s)\right) ds\right) \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\overline{H}(s)\,H(s)\,ds\right) \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}H'(s)\,ds\right) \qquad [\text{ using Fact 1 of appendix }] \\
&= \overline{H}(x)\,\exp\left(-H(x)\right) \\
&= \overline{H}(x)\,\exp\left(-\beta_0(\overline{H}(x))\right)
\end{aligned}$$

Now, assume that the assertion of the Lemma is true for $n \in \{1, 2, \ldots, k\}$, for some $k \ge 1$, then from definition we have

$$\begin{aligned}
f_{k+1}(x) &= T(f_k)(x) \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\left(\overline{H}(s) - f_k(s)\right) ds\right) \qquad [\text{ using (15) }] \\
&= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\overline{H}(s)\left(1 - e^{-\beta_{k-1}(\overline{H}(s))}\right) ds\right) \\
&= \overline{H}(x)\,\exp\left(-\int_{\overline{H}(x)}^{1}\frac{1}{w}\left(1 - e^{-\beta_{k-1}(1-w)}\right) dw\right)
\end{aligned} \tag{19}$$

The last equality follows by substituting $w = H(s)$ and thus from Fact 1 and Fact 2 of the appendix we get that $\frac{dw}{w} = \overline{H}(s)\,ds$ and $H(-x) = \overline{H}(x)$. Finally by definition of β_n's and using (19) we get $f_{k+1} = T(f_k)$.

∎

To complete the proof it is now enough to show that $\beta_n \to 0$ pointwise, which will imply by Lemma 6 that $f_n \to \overline{H}$ pointwise, as $n \to \infty$. Using Proposition 11 (see Section 4) we get the following characterization of the pointwise limit of these β_n's.

Lemma 7 *There exists a function* $L : [0,1] \to [0,1]$ *with* $L(1) = 0$*, such that*

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw, \ \forall\, s \in [0,1), \tag{20}$$

and $L(s) = \lim\limits_{n\to\infty} \beta_n(s), \ \forall\ 0 \le s \le 1.$

Proof : From part (b) of Proposition 11 we know that for any $s \in [0,1]$ the sequence $\{\beta_n(s)\}$ is decreasing, and hence $\exists$ a function $L : [0,1] \to [0,1]$ such that $L(s) = \lim\limits_{n\to\infty} \beta_n(s)$. Now observe that $\beta_n(1-w) \le \beta_0(1-w) = w, \ \forall\ 0 \le w \le 1$, and hence

$$0 \le \frac{1}{w}\left(1 - e^{-\beta_n(1-w)}\right) \le \frac{\beta_n(1-w)}{w} \le 1, \ \ \forall\ 0 \le w \le 1.$$

Thus by taking limit as $n \to \infty$ in (17) and using the *dominated convergence theorem* along with part (a) of Proposition 11 we get that

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw, \ \ \forall\ 0 \le s < 1.$$

∎

The above lemma basically translates the non-linear integral equation (12) to the non-linear integral equation (20), where the solution $g = \overline{H}$ of (12) is given by the solution $L \equiv 0$ of (20). So at first sight this may not lead us to the conclusion. But fortunately, something nice happens for equation (20), and we have the following result which is enough to complete the proof of Theorem 1.

Lemma 8 *If* $L : [0,1] \to [0,1]$ *is a function which satisfies the non-linear integral equation (20), namely,*

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw, \ \ \forall\ 0 \le s < 1,$$

and if $L(1) = 0$*, then* $L \equiv 0$*.*

Proof : First note that $L \equiv 0$ is a solution. Now let L be any solution of (20), then L is infinitely differentiable on the open interval $(0,1)$, by repetitive application of *Fundamental Theorem of Calculus.*

Consider,

$$\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1, \; w \in [0,1]. \tag{21}$$

Observe that $\eta(0) = \eta(1) = 0$ as $L(1) = 0$. Now, from (20) we get that

$$L'(w) = -\frac{1}{w}\left(1 - e^{-L(1-w)}\right), \; w \in (0,1). \tag{22}$$

Thus differentiating the function η we get

$$\eta'(w) = e^{-L(w)}\left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0, \; \forall \; w \in (0,1). \tag{23}$$

So the function η is decreasing in $(0,1)$ and is continuous in $[0,1]$ with boundary values as 0, hence $\eta \equiv 0 \Leftrightarrow L \equiv 0$.

■

4 Some Technical Details

In this section we prove some results which were used in Sections 2 and 3 for proving Theorems 1 and Corollary 3. These results are mainly technical details and hence have been omitted in the previous sections.

Proposition 9 *The operator Λ defined in Theorem 2 is weakly continuous when restricted to the subspace $\mathfrak{S}^\star$ as defined in (10).*

Proof : Let $\{\nu_n\}_{n=1}^\infty \subseteq \mathfrak{S}^\star$ and suppose that $\nu_n \xrightarrow{d} \nu \in \mathfrak{S}^\star$. We will show that $\Lambda(\nu_n) \xrightarrow{d} \Lambda(\nu)$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space such that, $\exists \; \{(X_n, Y_n)\}_{n=1}^\infty$ and (X,Y) random vectors taking values in $\mathbb{R}^2$, with $(X_n, Y_n) \sim \nu_n, \; n \geq 1$, and $(X,Y) \sim \nu$. Notice that by definition $X_n \stackrel{d}{=} Y_n \stackrel{d}{=} X \stackrel{d}{=} Y$, and each has Logistic distribution.

Fix $x, y \in \mathbb{R}$, then using similar calculations as in (11) we get

$$\begin{aligned} G_n(x,y) &:= \Lambda(\nu_n)\left((x,\infty) \times (y,\infty)\right) \\ &= \overline{H}(x)\overline{H}(y) \exp\left(-\int_0^\infty \mathbf{P}\left(X_n > t - x, Y_n > t - y\right) dt\right) \\ &= \overline{H}(x)\overline{H}(y) \exp\left(-\int_0^\infty \mathbf{P}\left((X_n + x) \wedge (Y_n + y) > t\right) dt\right) \\ &= \overline{H}(x)\overline{H}(y) \exp\left(-\mathbf{E}\left[(X_n + x)^+ \wedge (Y_n + y)^+\right]\right), \end{aligned} \tag{24}$$

and a similar calculation will also give that

$$\begin{aligned} G(x,y) &:= \Lambda(\nu)\left((x,\infty)\times(y,\infty)\right) \\ &= \overline{H}(x)\overline{H}(y)\,\exp\left(-\mathbf{E}\left[(X+x)^+\wedge(Y+y)^+\right]\right). \end{aligned} \tag{25}$$

Now to complete the proof all we need is to show

$$\mathbf{E}\left[(X_n+x)^+\wedge(Y_n+y)^+\right]\longrightarrow\mathbf{E}\left[(X+x)^+\wedge(Y+y)^+\right].$$

Since we assumed that $(X_n,Y_n)\ \stackrel{d}{\longrightarrow}\ (X,Y)$ thus

$$(X_n+x)^+\wedge(Y_n+y)^+\ \stackrel{d}{\longrightarrow}\ (X+x)^+\wedge(Y+y)^+\,,\ \ \forall\ x,y\in\mathbb{R}. \tag{26}$$

Fix $x,y\in\mathbb{R}$, define $Z_n^{x,y}:=(X_n+x)^+\wedge(Y_n+y)^+$, and $Z^{x,y}:=(X+x)^+\wedge(Y+y)^+$. Observe that

$$0\ \le\ Z_n^{x,y}\ \le\ (X_n+x)^+\le\ |X_n+x|\,,\ \ \forall\ \ n\ge 1. \tag{27}$$

But, $|X_n+x|\ \stackrel{d}{=}\ |X+x|\,,\ \forall\ n\ \ge\ 1$. So clearly $\{Z_n^{x,y}\}_{n=1}^{\infty}$ is uniformly integrable. Hence we conclude (using Theorem 25.12 of Billingsley [5]) that

$$\mathbf{E}\left[Z_n^{x,y}\right]\longrightarrow\mathbf{E}\left[Z^{x,y}\right].$$

This completes the proof.

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Proposition 10 *The operator T maps $\mathfrak{F}$ into $\mathfrak{F}$.*

Proof : First note that if $f\in\mathfrak{F}$, then by definition $T(f)(x)\ge\overline{H}^2(x),\ \forall\ x\in\mathbb{R}$. Next by definition of $\mathfrak{F}$ we get that $f\in\mathfrak{F}\Rightarrow f\preccurlyeq\overline{H}$, thus

$$\begin{aligned} &\int_{-x}^{\infty}f(s)\,ds\ \le\ \int_{-x}^{\infty}\overline{H}(s)\,ds,\ \forall\ x\in\mathbb{R} \\ \Rightarrow\ \ &T(f)(x)\ \le\ \overline{H}^2(x)\,\exp\left(\int_{-x}^{\infty}\overline{H}(s)\,ds\right)\ =\ \overline{H}(x),\ \forall\ x\in\mathbb{R} \end{aligned}$$

The last equality follows from (A1) (see Fact 3 of appendix). So,

$$\overline{H}^2(x)\le T(f)(x)\le\overline{H}(x),\ \ \forall\ \ x\in\mathbb{R}. \tag{28}$$

Now we need to show that for $f \in \mathfrak{F}$, $T(f)$ is a tail of a distribution. From the definition $T(f)$ is continuous (in fact infinitely differentiable). Further using (28) and the fact that $\overline{H}$ is a tail of a distribution we get that

$$\lim_{x \to \infty} T(f)(x) = 0, \quad \text{and} \quad \lim_{x \to -\infty} T(f)(x) = 1. \tag{29}$$

Finally let $x \leq y$ be two real numbers, then

$$\int_{-x}^{\infty} \left(\overline{H}(s) - f(s) \right) \, ds \leq \int_{-y}^{\infty} \left(\overline{H}(s) - f(s) \right) \, ds,$$

because $f \preccurlyeq \overline{H}$. Also $\overline{H}(x) \geq \overline{H}(y)$, thus using (15) we get

$$T(f)(x) \geq T(f)(y) \tag{30}$$

So using (28), (29), (30) we conclude that $T(f) \in \mathfrak{F}$ if $f \in \mathfrak{F}$.

■

Proposition 11 *The following are true for the sequence of functions $\{\beta_n\}_{n \geq 0}$ as defined in (17).*

(a) For every $n \geq 1$, $\lim_{s \to 0+} \beta_n(s)$ exists, and is given by

$$\int_0^1 \tfrac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)} \right) \, dw,$$

we will write this as $\beta_n(0)$.

(b) For every fixed $s \in [0, 1]$, the sequence $\{\beta_n(s)\}$ is decreasing.

Proof : (a) Note that for $n = 1$,

$$\beta_1(s) = \int_s^1 \frac{1}{w} \left(1 - e^{w} \right) \, dw, \ \forall \ s \in (0, 1],$$

Thus $\lim_{s \to 0+} \beta_1(s)$ exists and is given by

$$\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_0(1-w)} \right) \, dw.$$

Now we assume that the assertion is true for $n \in \{1, 2, \ldots, k\}$ for some $k \geq 1$, we will show that it is true for $n = k + 1$. For that note

$$\beta_{k+1}(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_k(1-w)} \right) \, dw, \ \forall \ s \in (0, 1].$$

But,

$$\begin{aligned}
& \lim_{w\to 0+} \frac{1}{w}\left(1-e^{-\beta_k(1-w)}\right) \\
= \quad & \lim_{w\to 0+} \frac{1-e^{-\beta_k(1-w)}}{\beta_k(1-w)} \times \frac{\beta_k(1-w)}{w} \\
= \quad & \lim_{w\to 0+} \frac{1}{w}\int_{1-w}^{1} \frac{1}{v}\left(1-e^{-\beta_{k-1}(1-v)}\right) dv \\
= \quad & 1-e^{-\beta_{k-1}(0)}
\end{aligned}$$

The last equality follows from *mean-value theorem* and the induction hypothesis. The rest follows from the definition.

(b) Notice that $\beta_0(s) = 1-s$ for $s \in [0,1]$, thus

$$\beta_1(s) = \int_s^1 \frac{1-e^{-w}}{w}\, dw < 1-s = \beta_0(s), \ \ \forall \ s \in [0,1].$$

Now assume that for some $n \geq 1$ we have $\beta_n(s) < \beta_{n-1}(s) < \cdots < \beta_0(s), \ \forall \ s \in [0,1]$, if we show that $\beta_{n+1}(s) < \beta_n(s), \ \forall \ s \in [0,1]$ then by induction the proof will be complete. For that, fix $s \in [0,1]$ then

$$\begin{aligned}
\beta_{n+1}(s) \quad & = \quad \int_s^1 \frac{1}{w}\left(1-e^{-\beta_n(1-w)}\right) dw \\
& < \quad \int_s^1 \frac{1}{w}\left(1-e^{-\beta_{n-1}(1-w)}\right) dw \\
& = \quad \beta_n(s)
\end{aligned}$$

Hence the proof of the proposition.

■

5 Final Remarks

5.1 Comments on the proof of Theorem 1

(a) Intuitively, a natural approach to show that the fixed-point equation $\Lambda(\nu) = \nu$ on $\mathfrak{S}$ has unique solution, would be to specify a metric ρ on $\mathfrak{S}$ such that the operator Λ becomes a contraction with respect to it. Unfortunately, this approach seems rather hard or may even be impossible. For this reason we have taken a complicated route of proving the bivariate uniqueness using analytic techniques similar to Eulerian recursion.

(b) Although at first glance it seems that the operator T as defined in (13) is just an analytic tool to solve the equation (12), it has a nice interpretation through Logistic RDE (6). Suppose $\mathfrak{A}$ is the operator associated with Logistic RDE, that is,

$$\mathfrak{A}(\mu) \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right), \tag{31}$$

where $\left(\xi_j\right)_{j \geq 1}$ are points of a Poisson point process of mean intensity 1 on $(0, \infty)$, and are independent of $\left(X_j\right)_{j \geq 1}$, which are i.i.d with distribution μ on $\mathbb{R}$. It is easy to check that the domain of definition of $\mathfrak{A}$ is the space

$$\mathcal{A} := \left\{ F \,\middle|\, F \text{ is a distribution function on } \mathbb{R} \text{ and } \int_0^\infty \overline{F}(s)\, ds < \infty \right\}. \tag{32}$$

Note that in probabilistic terminology the condition $\int_0^\infty \overline{F}(s)\, ds < \infty$ means $\mathbf{E}_F\left[X^+\right] < \infty$.

Notice that from definition $\mathfrak{F} \subseteq \mathcal{A}$, and T can be naturally extended to the whole of $\mathcal{A}$. In that case the following identity holds

$$\frac{\overline{T(\mu)}(\cdot)}{\overline{H}(\cdot)} \times \frac{\overline{\mathfrak{A}(\mu)}(\cdot)}{\overline{H}(\cdot)} = 1, \;\; \forall \;\, \mu \in \mathcal{A}. \tag{33}$$

This at least explains the monotonicity of T through anti-monotonicity property (easy to check) of the Logistic operator $\mathfrak{A}$. The identity (33) seems rather interesting and might be useful for deeper understanding of the bivariate uniqueness property of the Logistic fixed-point equation.

5.2 Domain of attraction

As discussed in Aldous and Bandyopadhyay [2], related to any fixed-point equation there is always the question of its domain of attraction. From the recursion proof we can clearly see that the equation (12) has the whole of $\mathfrak{F}$ within its domain of attraction. Thus it is natural to believe that one might be able to derive the uniqueness by a contraction argument.

It still remains an open problem to determine the exact domain of attraction of Logistic RDE. Unfortunately, the identity (33) does not seem to be useful in that regard.

5.3 Everywhere discontinuity of the operators Λ and $\mathfrak{A}$

From Proposition 9 we get that the operator Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ of its domain of definition, and we saw that this is enough regularity to conclude nice result like Corollary 3. It is still interesting to see if Λ is continuous on whole of its domain of definition. Unfortunately, it is just the opposite. Λ is discontinuous everywhere on its domain of definition. In fact, even the operator $\mathfrak{A}$ associated with the Logistic RDE as defined in (31), is discontinuous everywhere on its domain of definition $\mathcal{A}$. To see this we note that if $\mu \in \mathcal{A}$, and $\mu_n \xrightarrow{d} \mu$, where $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, then $\mathfrak{A}(\mu_n) \xrightarrow{d} \mathfrak{A}(\mu)$ provided $\mathbf{E}_{\mu_n}[X_n^+] \longrightarrow \mathbf{E}_{\mu}[X^+]$. Thus clearly $\mathfrak{A}$ is discontinuous everywhere on $\mathcal{A}$ and hence so is Λ.

Appendix

Here we provide some known basic facts about the Logistic distribution which are used in various places in the proofs.

First recall that we say a real valued random variable Z has Logistic distribution if its distribution function is given by (2), namely,

$$H(x) = \mathbf{P}(Z \leq x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

The following facts hold for the function H.

Fact 1 *H is infinitely differentiable, and $H'(\cdot) = H(\cdot)\overline{H}(\cdot)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$.*

Proof : From the definition it follows that H is infinitely differentiable on $\mathbb{R}$. Further,

$$\begin{aligned} H'(x) &= \frac{1}{1+e^{-x}} \times \frac{e^{-x}}{1+e^{-x}} \\ &= H(x)\,\overline{H}(x) \ \forall \ x \in \mathbb{R} \end{aligned}$$

■

Fact 2 *H is symmetric around 0, that is, $H(-x) = \overline{H}(x) \ \forall \ x \in \mathbb{R}$.*

Proof : From the definition we get that for any $x \in \mathbb{R}$,

$$H(-x) = \frac{1}{1+e^{x}} = \frac{e^{-x}}{1+e^{-x}} = \overline{H}(x).$$

■

Fact 3 $\overline{H}$ *is the unique solution of the non-linear integral equation*

$$\overline{H}(x) = \exp\left(- \int_{-x}^{\infty} \overline{H}(s)\, ds \right), \ \ \forall \ \ x \in \mathbb{R}. \tag{A1}$$

Proof : Notice that the equation (A1) is nothing but Logistic RDE, since $\mathfrak{A}(H)(x) = \exp\left(- \int_{-x}^{\infty} \overline{H}(s)\, ds \right), \ \forall \ x \in \mathbb{R}$ (see proof of Lemma 5 in Aldous [4]). Thus from the fact that $\overline{H}$ is the unique solution of Logistic RDE (Lemma 5 of Aldous [4]) we conclude that $\overline{H}$ is unique solution of equation (A1).

∎

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