On the distribution of ranked heights of excursions of a Brownian bridge *

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Abstract

The distribution of the sequence of ranked maximum and minimum values attained during excursions of a standard Brownian bridge $(B_t^{\rm br}, 0 \le t \le 1)$ is described. The height $M_j^{\rm br+}$ of the jth highest maximum over a positive excursion of the bridge has the same distribution as $M_1^{\rm br+}/j$, where the distribution of $M_1^{\rm br+} = \sup_{0\le t\le 1} B_t^{\rm br}$ is given by Lévy's formula $P(M_1^{\rm br+} > x) = e^{-2x^2}$. The probability density of the height $M_j^{\rm br}$ of the jth highest maximum of excursions of the reflecting Brownian bridge $(|B_t^{\rm br}|, 0 \le t \le 1)$ is given by a modification of the known θ-function series for the density of $M_1^{\rm br} = \sup_{0\le t\le 1} |B_t^{\rm br}|$. These results are obtained from a more general description of the distribution of ranked values of a homogeneous functional of excursions of the standardized bridge of a self-similar recurrent Markov process.

Keywords: Brownian bridge, Brownian excursion, Brownian scaling, local time, self-similar recurrent Markov process, Bessel process

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Contents

1 Introduction

Let $B^{\text{br}} := (B_t^{\text{br}}, 0 \le t \le 1)$ be a *standard Brownian bridge*, that is

$$(B_t^{\text{br}}, 0 \le t \le 1) \stackrel{d}{=} (B_t, 0 \le t \le 1 | B_1 = 0)$$

where $(B_t, t \ge 0)$ is a standard one-dimensional Brownian motion. See [35] for background. The random open subset $\{t : B_t^{\text{br}} \ne 0\}$ of $[0,1]$ is a countable union of maximal disjoint intervals (a,b), called *excursion intervals* of B^{br}, such that $B_a^{\text{br}} = B_b^{\text{br}} = 0$ and either $B_t^{\text{br}} > 0$ for all $t \in (a,b)$ (a *positive excursion interval*) or $B_t^{\text{br}} < 0$ for all $t \in (a,b)$ (a *negative excursion interval*). Let

$$M_1^{\text{br}+} \ge M_2^{\text{br}+} \ge \cdots > 0$$

be the ranked decreasing sequence of values $\sup_{t\in(a,b)} B_t^{\text{br}}$ obtained as (a,b) ranges over all positive excursion intervals of B^{br}. Similarly, let

$$M_1^{\text{br}-} \ge M_2^{\text{br}-} \ge \cdots > 0$$

the ranked values of $-\inf_{t\in(a,b)} B_t^{\text{br}}$ as (a,b) ranges over all negative excursion intervals of B^{br}, and let

$$M_1^{\text{br}} \ge M_2^{\text{br}} \ge \cdots > 0$$

be the ranked values of $\sup_{t\in(a,b)} |B_t^{\text{br}}|$ as (a,b) ranges over all excursion intervals of B^{br}. One motivation for study of the sequence (M_j^{br}) is that this sequence describes the asymptotic distribution as $n \to \infty$ of the ranked heights of tree components of the random digraph generated by a uniformly distributed random mapping of an n-element set to itself [1]. Note that

$$M_1^{\text{br}+} = \sup_{0\le t\le 1} B_t^{\text{br}}; \quad M_1^{\text{br}-} = -\inf_{0\le t\le 1} B_t^{\text{br}}; \quad M_1^{\text{br}} = \sup_{0\le t\le 1} |B_t^{\text{br}}| = M_1^{\text{br}+} \vee M_1^{\text{br}-}. \tag{1}$$

The main purpose of this paper is to describe as explicitly as possible the laws of the decreasing random sequences introduced above. In particular, we obtain the results stated in the following two theorems. Some of the results of this paper were presented without proof in [32].

Theorem 1 *For each $j = 1, 2, \ldots$ the common distribution of $M_j^{\text{br}+}$ and $M_j^{\text{br}-}$ is determined by the formula*

$$P(M_j^{\text{br}+} > x) = e^{-2j^2x^2} \qquad (x \ge 0) \tag{2}$$

while that of $M_j^{\rm br}$ *is determined by*

$$P(M_j^{\rm br} > x) = 2^j \sum_{n=0}^{\infty} \binom{-j}{n} e^{-2(n+j)^2 x^2} \quad (x \geq 0). \tag{3}$$

Formula (2) amounts to the identities in distribution

$$M_j^{\rm br+} \stackrel{d}{=} \frac{M_1^{\rm br+}}{j} \stackrel{d}{=} \frac{1}{j}\sqrt{\frac{\varepsilon}{2}} \tag{4}$$

for $j = 1, 2, \ldots$ where ε denotes a standard exponential variable. The second identity in (4) is Lévy's [25] well known description of the distribution of $\sup_{0\leq t\leq 1} B_t^{\rm br}$. Despite its simplicity, the first identity in (4) does not seem obvious without calculation. The case $j = 1$ of (3) is the well known Kolmogorov-Smirnov formula for the distribution of $M_1^{\rm br} = \sup_{0\leq t\leq 1} |B_t^{\rm br}|$, which arises in the asymptotic theory of empirical distribution functions [40, 12, 39, 26]:

$$P(M_1^{\rm br} > x) = 2\sum_{k=1}^{\infty}(-1)^{k-1} e^{-2k^2x^2} = 1 - \theta_3\left(\tfrac{\pi}{2}, \tfrac{2}{\pi} i x^2\right) \tag{5}$$

where

$$\theta_3(z,t) := \sum_{n=-\infty}^{\infty} e^{i\pi n^2 t}\cos(2nz)$$

is the classical Jacobi theta function defined for $t = \Re t + i\Im t \in \mathbb{C}$ with $\Im t > 0$. Formula (3) shows there is no relation as simple as (4) between the distribution of $M_j^{\rm br}$ for $j > 1$ and that of $M_1^{\rm br}$.

Define the intensity measure ν_M for the sequence $(M_j^{\rm br})$ by

$$\nu_M(A) = E\sum_{j=1}^{\infty} 1(M_j^{\rm br} \in A)$$

for Borel subsets A of $(0,\infty)$, and define ν_{M^+} similarly in terms of $(M_j^{\rm br+})$. Formula (2) implies that these intensity measures ν_M and ν_{M^+} are given by the formula

$$\nu_M(x,\infty) = 2\nu_{M^+}(x,\infty) = 2\sum_{j=1}^{\infty} e^{-2j^2x^2} = \theta_3\left(0, \tfrac{2}{\pi} i x^2\right) - 1 = \theta\left(\tfrac{2}{\pi}x^2\right) - 1 \tag{6}$$

where for $t > 0$

$$\theta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \theta_3(0, it). \tag{7}$$

Note the striking parallel between formulae (5) and (6). We now explain how formula (6) is related to the formula of Chung [7] for the distribution of the maximum M_* of a standard Brownian excursion, that is

$$P(M_* \leq x) = \theta(\tfrac{2}{\pi}x^2) + \tfrac{4}{\pi}x^2\theta'(\tfrac{2}{\pi}x^2) \qquad (x > 0) \tag{8}$$

where θ' is the derivative of θ. Riemann [36] gave the formula

$$\frac{1}{2}s(s-1)\int_0^\infty t^{\frac{s}{2}-1}(\theta(t)-1)dt = 2\xi(s) := s(s-1)\pi^{-s/2}, \, (s/2)\sum_{n=1}^\infty \frac{1}{n^s} \qquad (\Re s > 1) \tag{9}$$

and deduced from it and the classical functional equation

$$\theta(t) = t^{-1/2}\theta(t^{-1}) \qquad (t > 0)$$

that (9) defines a unique entire function ξ which satisfies the functional equation

$$\xi(s) = \xi(1-s) \qquad (s \in \mathbb{C}).$$

As shown by Biane-Yor [4], Chung's formula (8) for $P(M_* \leq x)$ is equivalent to the following expression of the Mellin transform of M_*:

$$E[M_*^s] = \left(\frac{\pi}{2}\right)^{\frac{s}{2}} 2\xi(s) \qquad (s \in \mathbb{C}). \tag{10}$$

See also [45, 3] for reviews of this circle of ideas and other interpretations of $\theta(t)$ in the context of Brownian motion.

These descriptions of the distribution of M_* are related to our description (6) of the intensity measure ν_M for the sequence (M_j^{br}) via the known result [42, 30] that the intensity measure for the lengths of excursions of the bridge

$$\nu_V(A) := E\sum_{j=1}^\infty 1(V_j^{\mathrm{br}} \in A),$$

where V_j^{br} is the length of the jth longest interval component of the random subset $\{t : B_t^{\mathrm{br}} \neq 0\}$ of $[0, 1]$, is determined by the density

$$\frac{\nu_V(dv)}{dv} = \frac{1}{2v^{3/2}} \qquad (0 < v < 1). \tag{11}$$

Indeed, by conditioning on the lengths of all the excursions of the Brownian bridge, and using (11), the intensity measure ν_M has density at $x > 0$

$$\frac{\nu_M(dx)}{dx} = \int_0^1 \frac{dv}{2v^2} f(x/\sqrt{v}) \quad \text{where} f(y) := \frac{P(M_* \in dy)}{dy}$$

For $p > 1$ this yields

$$\begin{aligned} \int_0^\infty x^p \nu_M(dx) &= \frac{E(M_*^p)}{(p-1)} \qquad \text{by Fubini} \\ &= \left(\frac{\pi}{2}\right)^{p/2} \frac{2\xi(p)}{(p-1)} \quad \text{by (10)} \\ &= \left(\frac{\pi}{2}\right)^{p/2} \int_0^\infty x^p (2x\theta'(x^2))dx \end{aligned}$$

where the last equality is a simple transformation of (9). Hence by uniqueness of Mellin transforms

$$\frac{\nu_M(dx)}{dx} = -\frac{4x}{\pi}\theta'(\tfrac{2}{\pi}x^2) \tag{12}$$

which is equivalent to (6). This calculation allows any one of the three formulae (6), (10) and (11) to be deduced from the other two.

Theorem 1 will be derived as a consequence of the next theorem, which characterizes the law of the entire sequence $(M_j^{\rm br})$. This result, along with a corresponding description of the law of $(M_j^{\rm br+})$, will be seen to be an expression of the fact that when each term in one of these sequences is multiplied by a suitable independent random factor, the result is the sequence of points of a simple mixture of Poisson processes. This key property is a consequence of the Poisson structure of excursions of Brownian motion combined with Brownian scaling.

Theorem 2 *Let N be a standard Gaussian variable independent of the Brownian bridge $B^{\rm br}$. Then the sequence $(|N|M_j^{\rm br}, j = 1, 2, \ldots)$ is Markovian, with one-dimensional distributions given by*

$$P(|N|M_j^{\rm br} \geq x) = (1 - \tanh x)^j \qquad (x \geq 0, j = 1, 2, \ldots)$$

and inhomogeneous transition probabilities

$$P(|N|M_j^{\rm br} \leq x \,|\, |N|M_{j-1}^{\rm br} = y) = \left(\frac{\tanh x}{\tanh y}\right)^j \qquad (0 \leq x \leq y, j = 1, 2, \ldots)$$

Despite the simple structure of the sequence $(|N|M_j^{\rm br}, j = 1, 2, \ldots)$ exposed by this result, the finite-dimensional distributions of the sequence $(M_j^{\rm br})$ appear to be rather complicated. This is yet another instance where the introduction of a suitable random multiplier provides a substantial simplification, as we have recognized in a number of other studies of homogeneous functionals of Brownian motion and self-similar Markov processes ([35, Ch. XII Ex. (4.24)],[6, 34]). See also Perman-Wellner [26] and Jansons [17] for further applications of this device.

The rest of this paper is organized as follows. In Section 2 we present a general characterization of the distribution of ranked values of a homogeneous functional F of excursions of the standardized bridge of a self-similar recurrent Markov process. We choose to work at this level of generality in order to expose the basic structure underlying both our previous work on ranked lengths of excursions [28, 30] and our present study of ranked heights. Section 3 shows how the general results of Section 2 may be applied to the heights of excursions of a Brownian bridge to obtain Theorems 1 and 2. Section 4 indicates how these results for a Brownian bridge may be generalized to the bridge of a recurrent Bessel process. In Sections 5, 6 and 7 we return to the general setting of Section 2 to consider excursions of the basic Markov process up to an inverse local time. In particular Section 6 presents some generalizations of results of Biane-Yor [4] regarding the maximum of a Brownian or Bessel excursion, and Section 7 generalizes some of the results of Knight [21] and Pitman-Yor [30]. See also [8, 9] for some further applications of the results of this paper.

2 Bridges and Excursions of a self-similar Markov process.

Recall that for $\beta \in \mathbb{R}$ a process $B := (B_t, t \geq 0)$ is called *β-self-similar* [23, 24, 41, 38] if B has the *scaling property*

$$(B_{ct}, t \geq 0) \stackrel{d}{=} (c^\beta B_t, t \geq 0) \text{ for each } c > 0, \tag{13}$$

which generalizes the well known scaling property of Brownian motion for $\beta = \frac{1}{2}$. We sometimes write $B(t)$ instead of B_t for typographical convenience. Suppose in this section that B is a real or vector-valued β-self-similar strong-Markov process, with starting state 0 which is a recurrent point for B. It is well known that B then has a continuous increasing local time process at 0, say $(L_t, t \geq 0)$, whose inverse process $(\tau_\ell, \ell \geq 0)$ is a stable subordinator of index α for some $0 < \alpha < 1$. That is to say (τ_ℓ) is an increasing

α^{-1}-self-similar process with stationary independent increments, hence

$$E \exp(-\lambda \tau_\ell) = \exp(-\ell K \lambda^\alpha) \tag{14}$$

for some constant $K > 0$.

For $t > 0$ let $G_t := \sup\{s : s \leq t, B_s = 0\}$ be the last zero of B before time t and let $D_t := \inf\{s : s \geq t, B_s = 0\}$ be the first zero of B after time t. It follows easily from the scaling property of B that for any fixed time $T > 0$, and hence for any random time T which is independent of B, the process

$$B_T^{\mathrm{br}}(u) := G_T^{-\beta} B(uG_T) \qquad (0 \leq u \leq 1) \tag{15}$$

has a distribution which does not depend on the choice of T. Call a process with this distribution a *standard B-bridge*, denoted B^{br}. Intuitively,

$$B^{\mathrm{br}} \stackrel{d}{=} (B_u, 0 \leq u \leq 1 \,|\, B_0 = B_1 = 0).$$

Similar remarks apply to the process

$$B_T^{\mathrm{ex}}(u) := (D_T - G_T)^{-\beta} B(G_T + u(D_T - G_T)) \qquad (0 \leq u \leq 1) \tag{16}$$

whose distribution defines that of a *standard B-excursion*, denoted B^{ex}. Intuitively,

$$B^{\mathrm{ex}} \stackrel{d}{=} (B_u, 0 \leq u \leq 1 \,|\, B_0 = B_1 = 0, B_u \neq 0 \text{ for } 0 < u < 1).$$

See [14, 29] and papers cited there for more about Markovian bridges and excursions.

Let $(e_t, 0 \leq t \leq V_e)$ denote a generic excursion path, where V_e is the *lifetime* or *length* of e. Let F be a non-negative measurable functional of excursions e, and let $\gamma > 0$. Call F a *γ-homogeneous functional of excursions of B* if

$$F(e_t, 0 \leq t \leq V_e) = V_e^\gamma F(V_e^{-\beta} e_{uV_e}, 0 \leq u \leq 1). \tag{17}$$

In particular, we have in mind the following functionals F: length, maximum height, maximum absolute height, area, maximum local time .... About areas, see [27] and papers cited there.

Theorem 3 *Let F be a γ-homogeneous functional of excursions of B, let $F_* := F(B^{\mathrm{ex}})$ for a standard excursion B^{ex}, and suppose that $E((F_*)^{\alpha/\gamma}) < \infty$, for α as in (14). Then* *the strictly positive values of $F(e)$ as e ranges over the countable collection of excursions of B^{br} can be arranged as a sequence*

$$F_1^{\mathrm{br}} \geq F_2^{\mathrm{br}} \geq \cdots > 0.$$

Let $,_{\alpha}$ be a random variable, independent of B^{br}, with the gamma(α) density

$$P(,_{\alpha} \in dt)/dt = ,(\alpha)^{-1} t^{\alpha-1} e^{-t} \qquad (t > 0). \tag{18}$$

Fix $\lambda > 0$. Then the joint distribution of the sequence $(F_j^{\mathrm{br}}, j = 1, 2, \ldots)$ is uniquely determined by the equality in distribution

$$(\mu(\lambda^{-\gamma},{}_{\alpha}^{\gamma} F_j^{\mathrm{br}}), j = 1, 2, \ldots) \stackrel{d}{=} (T_j^*, j = 1, 2, \ldots) \text{ where } T_j^* := \sum_{i=1}^{j} \varepsilon_i / \varepsilon_0 \tag{19}$$

for independent standard exponential variables $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$, and μ is the function determined as follows by α, λ, γ and the distribution of F_*:

$$\mu(x) := \int_0^\infty \frac{\alpha \lambda^{-\alpha}}{,(1-\alpha)} t^{-\alpha-1} e^{-\lambda t} P[\, F_* > x t^{-\gamma}\,]\, dt. \tag{20}$$

Remarks. In what follows we sometimes write $\mu_\lambda(x)$ instead of $\mu(x)$ to emphasize the dependence of this function on λ. It is evident from either (20) or (19) that $\mu_\lambda(x) = \mu_1(x\lambda^\gamma)$. The function μ plays a central role throughout the paper. Some alternative formulae for μ are presented in Corollaries 5 and 11. For the moment, we just note using (20) and $e^{-\lambda t} \leq 1$ that

$$\mu_\lambda(x) \leq \frac{\lambda^{-\alpha}}{,(1-\alpha)} E\left[(F_*/x)^{\alpha/\gamma} \right] < \infty \tag{21}$$

under the hypotheses of the theorem. The following lemma presents some preliminaries for the proof of the theorem.

Lemma 4 *Let T_λ be an exponential random variable with rate λ, assumed independent of B. Then*
(i) *the value L_{T_λ} of the local time process at 0 at time T_λ has an exponential distribution with rate $K\lambda^\alpha$;*
(ii) *Let $B^{\mathrm{br}} := (B_{T_\lambda}^{\mathrm{br}}(u), 0 \leq u \leq 1)$ be the bridge constructed as in (15) by rescaling the path B over $[0, G_{T_\lambda}]$. Then G_{T_λ} is independent of B^{br}, and $G_{T_\lambda} \stackrel{d}{=} \lambda^{-1},_{\alpha}$.*

Proof. To check (i), use

$$P(L_{T_\lambda} > \ell) = P(\tau_\ell < T_\lambda) = E\exp(-\lambda\tau_\ell) = \exp(-\ell K\lambda^\alpha).$$

The facts in (ii) are well known for B a Brownian motion or Bessel process. See e.g. [2] and papers cited there. As their proofs rely only on the self-similarity and strong Markov properties of B, they apply in the present setting too. □

Proof of Theorem 3. For $t > 0$ let

$$(F_j(t), j = 1, 2, \ldots) \tag{22}$$

denote the ranked sequence of strictly positive values of $F(e)$ obtained as e ranges over the countable collection of excursions completed by B during the interval $[0, t]$. As discussed in detail in Proposition 10 below, this sequence is a.s. well defined for $t = \tau_\ell$ for any $\ell > 0$, under the assumption that $E((F_*)^{\alpha/\gamma}) < \infty$. It follows easily that the sequence is well defined for all $t \geq 0$ a.s., hence in particular for $t = G_{T_\lambda}$. Using part (ii) of Lemma 4, (15), and the scaling property (17) of F we find that

$$(\lambda^{-\gamma}, {}_{\alpha}^{\gamma}F_j^{\mathrm{br}}, j = 1, 2, \ldots) \stackrel{d}{=} (F_j(G_{T_\lambda}), j = 1, 2, \ldots). \tag{23}$$

Now fix λ, and to simplify the rest of the proof let the local time process be normalized so that $K\lambda^\alpha = 1$. According to part (i) of Lemma 4, this makes

$$L_{T_\lambda} \stackrel{d}{=} \varepsilon_0. \tag{24}$$

By application of Itô's excursion theory, conditionally given $L_{T_\lambda} = \ell$ the strictly positive $F_j(G_{T_\lambda})$ are the ranked points of a PPP($\ell\widetilde{\mu}$), that is a Poisson point process with intensity measure $\ell\widetilde{\mu}$, where $\widetilde{\mu}$ is the measure on $(0, \infty)$ whose mass on (x, ∞) is the function $\mu(x)$ defined by (20). To briefly interpret the different factors appearing in formula (20), the measure

$$\frac{\alpha\lambda^{-\alpha}}{,\,(1-\alpha)} t^{-\alpha-1} dt$$

on $(0, \infty)$ is the Lévy measure governing the PPP of jumps of the subordinator (τ_ℓ); these jumps form the lengths of excursions of B. The factor $e^{-\lambda t}$ is the chance that an excursion of length t survives exponential killing at rate λ. The factor $P(F(B^{\mathrm{ex}}) > xt^{-\gamma})$ is the chance, given that an excursion has length t, that its F-value exceeds x. See for instance [16, 37, 28] for details of similar arguments. It follows that

$$(\mu(F_j(G_{T_\lambda})), j = 1, 2, \ldots \mid L_{T_\lambda} = \ell) \stackrel{d}{=} (,\,{}_j/\ell, j = 1, 2, \ldots) \tag{25}$$

where the $\Gamma_j := \sum_{i=1}^{j} \varepsilon_i$ are the points of a homogeneous Poisson process on $(0, \infty)$ with rate 1. Now (23), (24) and (25) combine to give (19). To see that the distribution of $(F_j^{\mathrm{br}}, j = 1, 2, \ldots)$ is uniquely determined by (19), it is enough to recover from (19) the distribution of $\Sigma := \sum_{j=1}^{k} \alpha_j F_j^{\mathrm{br}}$ for arbitrary non-negative α_j and $k = 1, 2, \ldots$. But formula (19) determines the distribution of $\Sigma\Gamma_\alpha^\gamma$, hence that of $\log\Sigma + \gamma \log\Gamma_\alpha$, where Σ is independent of Γ_α. But this in turn determines $E\exp(it\log\Sigma)$ for all real t, hence the distribution of Σ, because $E\exp(it\gamma\log\Gamma_\alpha)$ does not vanish for any real t, due to the infinite divisibility of $\log\Gamma_\alpha$ which follows from Gordon's representation [15] of $\log\Gamma_\alpha$ as an infinite sum of independent centered exponential variables. □

Corollary 5 *The function $\mu_\lambda(x)$ defined by (20) is also determined by the following formula:*

$$\exp[-K\lambda^\alpha(1 + \mu_\lambda(x))] = E[\exp(-\lambda\tau_1)1(F_1(\tau_1) \le x)] \tag{26}$$

where $F_1(\tau_1)$ is the largest value of $F(e)$ as e ranges over the excursions of B completed by the inverse local time τ_1, and K is determined by $\exp(-K\lambda^\alpha) = E[\exp(-\lambda\tau_1)]$.

Proof. As in the proof of Theorem 3, interpret $K\lambda^\alpha\mu(x)$ as the rate per unit local time of excursions e with $F(e) > x$, with an excursion of length t counted only with the probability $e^{-\lambda t}$ that it survives killing at rate λ. From the definition of K, the rate of killed excursions is $K\lambda^\alpha$. So $K\lambda^\alpha(\mu(x) + 1)$ is the rate of excursions e such that either $F(e) > x$ or e is killed. Formula (26) displays two expressions for the probability that there is no excursion e in the interval $[0, \tau_1]$ such that either $F(e) > x$ or e is killed. The first expression derives from the Poisson character of the excursion process, and the second is obtained by conditioning on $(\tau_1, F_1(\tau_1))$. □

The next lemma gives another description of the distribution of the sequence (T_j^*) appearing in formula (19).

Lemma 6 *For a sequence of random variables $(T_j^*, j = 1, 2, \ldots)$ the following two conditions are equivalent:*
(i) $T_j^* = \sum_{i=1}^{j} \varepsilon_i/\varepsilon_0$ *for independent standard exponential variables $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$;*
(ii) *The sequence is Markovian with one-dimensional distributions*

$$P(T_j^* \le x) = \left(\frac{x}{1+x}\right)^j \qquad (x \ge 0) \tag{27}$$

and inhomogeneous transition probabilities

$$P(T_j^* > y \,|\, T_{j-1}^* = x) = \left(\frac{x+1}{y+1}\right)^j \qquad (0 \le x \le y, \quad j = 2, 3, \ldots). \tag{28}$$

Proof. Suppose (i). Then (27) is elementary, and (28) can be seen as follows. Regard the (T_j^*) as arrival times in a mixed Poisson process obtained by assigning a standard exponential distribution to the arrival rate of a homogeneous Poisson process. Let

$$N^*(t) := \sum_{j=1}^{\infty} 1(T_j^* \leq t) \qquad (t \geq 0)$$

be the associated counting process. It is known [10, p. 532] that this process is an inhomogeneous Markov counting process with transition intensity $(n+1)/(t+1)$ for a jump up by 1 at time $t+$ given $N^*(t) = n$. Let

$$N^{\text{birth}}(v) := 1 + N^*(e^v - 1), \qquad v \geq 0.$$

Then it is easily verified that N^{birth} is a Yule process with birth rate 1, that is a homogeneous Markov counting process with intensity n for a jump up at time $v+$ given $N^{\text{birth}}(v) = n$, started at $N^{\text{birth}}(0) = 1$. Hence

$$T_0^* := 0 \text{ and } T_j^* = \exp\left(\sum_{k=1}^{j} \frac{\tilde{\varepsilon}_k}{k}\right) - 1 \quad (j = 1, 2, \ldots),$$

where the $\tilde{\varepsilon}_k := k \log((T_k^* + 1)/(T_{k-1}^* + 1))$ are independent standard exponentials, and (28) follows. Thus (i) implies (ii). Still assuming (i), it follows from the strong law of large numbers that $\lim_j T_j^*/j = \varepsilon_0^{-1}$ a.s., which allows ε_0 and hence the ε_i to be recovered a.s. as product measurable functions of the sequence (T_j^*). Since condition (ii) determines the distribution of the sequence (T_j^*), it follows immediately that (ii) implies (i). $\square$

The particular mixed Poisson process involved here has been studied and applied in a number of contexts [10]. The connection between this mixed Poisson process and the Yule process $(N^{\text{birth}}(v), v \geq 0)$, exploited above, amounts to the result of Kendall [18, Theorem 1] that the Yule process conditioned on $\varepsilon_0 := \lim_{v\to\infty} e^{-v} N^{\text{birth}}(v)$ is an inhomogeneous Poisson process. By combining Theorem 3 and Lemma 6, we immediately obtain:

Corollary 7 *Fix $\lambda > 0$. In the setting of Theorem 3, the sequence*

$$\widehat{F}_j^{\text{br}} := \lambda^{-\gamma}, {}^{\gamma}_{\alpha} F_j^{\text{br}} \qquad (j = 1, 2, \ldots)$$

is Markovian with one dimensional distributions

$$P(\widehat{F}_j^{\text{br}} \geq w) = \left(\frac{\mu(w)}{1 + \mu(w)}\right)^j \qquad (w > 0,\ j = 1, 2, \ldots) \tag{29}$$

and inhomogeneous transition probabilities

$$P(\widehat{F}_j^{\mathrm{br}} \leq w \mid \widehat{F}_{j-1}^{\mathrm{br}} = z) = \left(\frac{\mu(z)+1}{\mu(w)+1}\right)^j \qquad (0 \leq w \leq z, j = 2, 3\ldots) \tag{30}$$

where μ *is the function defined by* (20)*.*

Corollary 7 provides a very explicit description of the finite dimensional distributions of the sequence $(\widehat{F}_j^{\mathrm{br}})$ in terms of the basic function μ. The joint density of any finite number of consecutive terms is available, and the dependence structure is simple. This description determines in principle the finite-dimensional distributions of the sequence (F_j^{br}). Some features of this sequence, such as the distribution of a ratio of terms, can be read directly from the two-dimensional distributions of the sequence $(\widehat{F}_j^{\mathrm{br}})$. Other features are harder to obtain explicitly.

Since $E(\Gamma_\alpha^{\gamma p}) = \Gamma(\gamma p + \alpha)/\Gamma(\alpha)$, we find from (29) with $\lambda = 1$ that for $j = 1, 2, \ldots$

$$E(F_j^{\mathrm{br}})^p = \frac{\Gamma(\alpha)}{\Gamma(\gamma p + \alpha)} \int_0^\infty p x^{p-1} \left(\frac{\mu_1(x)}{1+\mu_1(x)}\right)^j dx \quad (p > 0). \tag{31}$$

Moreover the distribution of F_j^{br} is determined by this Mellin transform in p provided it is finite for p in some open interval. The joint distribution of the (F_j^{br}) is not so easy to describe. In particular, the sequence (F_j^{br}) does not necessarily have the Markov property, as shown by the following example. We presume that (F_j^{br}) is typically not Markovian, but do not see how to formulate a precise result to this effect.

Example 8 *Lengths of excursions.* Let V_j^{br} be the length of the jth longest excursion of B^{br}. So $V_j^{\mathrm{br}} = F_j^{\mathrm{br}}$ for $F(e) = V(e)$ the length of excursion e. In this case $\gamma = 1$, $F(B^{\mathrm{ex}}) = 1$, and taking $\lambda = 1$, formula (20) becomes

$$\mu(x) := \int_x^\infty \frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha-1} e^{-t} dt \tag{32}$$

Since $\sum_j V_j^{\mathrm{br}} = 1$ almost surely, we find from (19) that

$$(V_j^{\mathrm{br}};\, j = 1, 2, \ldots) \stackrel{d}{=} (\mu^{-1}(T_j^*)/\Sigma;\, j = 1, 2, \ldots) \tag{33}$$

where $\Sigma := \sum_j \mu^{-1}(T_j^*)$. The distribution of the sequence (V_j^{br}) is the particular case $\alpha = \theta$ of the Poisson-Dirichlet (PD) distribution with two parameters (α, θ), which was studied in [30]. The formula for moments $V_j^{\rm br}$ given by (31) in this case is a simplification of the instance $\alpha = \theta$ of Proposition 17 of [30] for PD(α, θ). In the notation of [30], the function $\mu(x)$ in (32) is $\mu(x) = \phi_\alpha(x)/(x^\alpha, (1-\alpha))$. The simplification occurs by an integration by parts, using the Wronskian identity displayed in equation (87) of [30]. The construction of a sequence with PD(α, α) distribution displayed on the right side of (33) is new. This representation is reminiscent of the representation of a PD$(\alpha, 0)$ distributed sequence given in Corollary 9 of [30]. But we do not know of any corresponding construction of PD(α, θ) for general $\theta > -\alpha$. Note that the sequence $(V_j^{\rm br})$ is not Markovian, because given $V_1^{\rm br}$ and $V_2^{\rm br}$ the next term $V_3^{\rm br}$ is subject to the constraint $V_3^{\rm br} \leq 1 - V_1^{\rm br} - V_2^{\rm br}$. While various explicit descriptions of the finite dimensional distributions of the sequence $(V_j^{\rm br})$ can be read from results of [30], it is not evident from any of these descriptions why multiplying the sequence $(V_j^{\rm br})$ by an independent gamma(α) variable yields a Markovian sequence.

3 Proofs of Theorems 1 and 2

Let B be a one-dimensional Brownian motion. Consider first

$$F(e) := M(e) := \sup_{0 \leq t \leq V_e} |e_t|, \tag{34}$$

the maximum absolute value attained by excursion e, and write $M_j(t)$ instead of $F_j(t)$. Then $M_1(t) = \sup_{0 \leq u \leq t} |B_u|$ for $t > 0$. According to a well known result of Knight [20], which is derived from the perspective of Itô's excursion theory in [33], for $\ell, x > 0$ there is the formula

$$\log E[\exp(-\lambda \tau_\ell) 1(M_1(\tau_\ell) \leq x)] = -\ell \sqrt{2\lambda} \coth(\sqrt{2\lambda} x) \tag{35}$$

where (L_t) is normalized as the occupation density of B at 0 relative to Lebesgue measure, so $L_t \stackrel{d}{=} |B_t|$ for each fixed t, and $K = \sqrt{2}$. Combine (26) and (35) to see that for B a Brownian motion, $F = M$, $\alpha = \gamma = \lambda = \frac{1}{2}$, the function $\mu(x)$ in Theorem 3 can be evaluated as

$$\mu(x) = \coth(x) - 1 = 2/(e^{2x} - 1). \tag{36}$$

Since $\gamma = \alpha = \lambda = \frac{1}{2}$ there is the identity in distribution $\lambda^{-\gamma}, \frac{\gamma}{\alpha} \stackrel{d}{=} |N|$ where $N \stackrel{d}{=} B_1$ has standard normal distribution. So for N independent of $B^{\rm br}$ the identity (29) gives for $x > 0$

$$P(|N| M_j^{\rm br} \geq x) = \left(\frac{2/(e^{2x} - 1)}{1 + 2/(e^{2x} - 1)} \right)^j = \left(\frac{2}{e^{2x} + 1} \right)^j \tag{37}$$

and we deduce Theorem 2. If instead we take

$$F(e) := M^+(e) := \sup_{0 \le t \le V_e} e_t \tag{38}$$

then $\mu(x)$ in (36) is replaced by $\frac{1}{2}\mu(x)$. This is clear from (20), because the sign of B^{ex} is positive with probability $\frac{1}{2}$ and independent of $M(B^{\text{ex}})$. Thus for $x > 0$

$$P(|N| M_j^{\text{br}+} \ge x) = \left(\frac{1/(e^{2x} - 1)}{1 + 1/(e^{2x} - 1)} \right)^j = e^{-2jx}. \tag{39}$$

It is known [43] that for ε a standard exponential variable independent of N

$$|N| \sqrt{\frac{\varepsilon}{2}} \stackrel{d}{=} \tfrac{1}{2}\varepsilon. \tag{40}$$

Compare (40) and (39) to deduce (2), first for $j = 1$, then for $j > 1$ by scaling. This argument appeals to the *uniqueness of the* $|N|$*-transform*, meaning that for positive random variables X and Y, each independent of $|N|$,

$$|N| X \stackrel{d}{=} |N| Y \text{ implies } X \stackrel{d}{=} Y \tag{41}$$

as justified in the proof of Theorem 3. To derive formula (3), expand the right side of (37) as

$$P(|N| M_j^{\text{br}} \ge x) = 2^j e^{-2jx} (1 + e^{-2x})^{-j} = 2^j \sum_{n=0}^{\infty} \binom{-j}{n} e^{-2(n+j)x}. \tag{42}$$

If we regard the $|N|$-transform as a linear operator on measures on $(0, \infty)$, say $m \to \tilde{m}$, and trust that this transform has reasonable properties, then formula (3) becomes evident as follows. Denote by P_k^+ the distribution of $M_k^{\text{br}+}$ on $(0, \infty)$. According to (37), (42) and (2) the $|N|$-transform of the distribution P_j of M_j^{br} is $\tilde{P}_j = \sum_n c_{n,j} \tilde{P}_{n+j}^+$ for some coefficients $c_{n,j}$. So it is reasonable to expect $P_j = \sum_n c_{n,j} P_{n+j}^+$, as asserted in (3). To make this argument rigorous, it seems necessary to establish a uniqueness result for the $|N|$-transform regarded as an operator on an appropriate class of signed measures, and to justify switching the order of the $|N|$-transform and an infinite summation. Rather than that, we finish the argument by appealing instead to the underlying probabilistic relationship between the two sequences (M_j^{br}) and $(M_j^{\text{br}+})$. Due to the independence of signs and absolute heights of excursions of B^{br},

$$M_j^{\text{br}+} = M_{H_j}^{\text{br}}; \qquad M_j^{\text{br}-} = M_{T_j}^{\text{br}} \qquad (j = 1, 2, \ldots) \tag{43}$$

where H_j is the index of the jth head and T_j the index of the jth tail in a sequence of independent fair coin tosses which is independent of (M_j^{br}). From (43) we obtain

$$P(M_j^{\mathrm{br}+} > x) = \sum_{h=1}^{\infty} P(H_j = h) P(M_h^{\mathrm{br}} > x). \tag{44}$$

Let $k := j - 1$ and $m := h - 1$ to deduce from (2) and the negative binomial distribution of H_j that for $k = 0, 1, 2, \ldots$

$$e^{-2(k+1)^2 x^2} = \sum_{m=0}^{\infty} 2^{-m-1} \binom{m}{k} P(M_{m+1}^{\mathrm{br}} > x)$$

This relation can be inverted to yield (3) by application of the following lemma to the sequences

$$b_k := e^{-2(k+1)^2 x^2}; \qquad a_m := 2^{-m-1} P(M_{m+1}^{\mathrm{br}} > x)$$

for an arbitrary fixed $x > 0$.

Lemma 9 *Let*

$$b_k := \sum_{m=0}^{\infty} \binom{m}{k} a_m \qquad (k = 0, 1, \ldots)$$

be the binomial moments of a non-negative sequence $(a_m, m = 0, 1, \ldots)$. Let $B(\theta) := \sum_{k=0}^{\infty} b_k \theta^k$ and suppose $B(\theta_1) < \infty$ for some $\theta_1 > 1$. Then

$$a_m = \sum_{k=0}^{\infty} (-1)^{k-m} \binom{k}{m} b_k \qquad (m = 0, 1, \ldots)$$

where the series is absolutely convergent.

Proof. Let $A(z) := \sum_{m=0}^{\infty} a_m z^m$. Then $k! b_k$ is the kth derivative of $A(z)$ evaluated at $z = 1$, hence $A(1 + \theta) = B(\theta)$ provided $|\theta| < \theta_1$. It follows that for $|z| < \theta_1 - 1$

$$A(z) = B(z-1) = \sum_{k=0}^{\infty} b_k (z-1)^k = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^{k-m} \binom{k}{m} b_k \right) z^m$$

where the series is absolutely convergent by consideration of $A(2 + z) = B(z + 1)$. $\Box$

4 Applications to Bessel processes.

We first recall from [33] a generalization of Knight's formula (35). For P^0 the expectation operator governing a recurrent diffusion process X on $[0, \infty)$ started at 0:

$$P^0[\exp(-\lambda\tau_1)1(M_1(\tau_1) \leq x)] = \exp[-(g_\lambda(0) - P^0(e^{-\lambda T_x})g_\lambda(x))^{-1}] \tag{45}$$

where $M_1(\tau_1)$ is the maximum of X up to τ_1, an inverse local time at 0, the local time process is normalized as occupation density relative to m, a constant multiple of the speed measure of X,

$$g_\lambda(x) := \int_0^\infty dt\, e^{-\lambda t} P(X_t \in dx)/m(dx)$$

is the λ-potential density of X relative to m, and T_x is the hitting time of x by X. We now take for X a self-similar diffusion B. As shown by Lamperti [24], self-similarity of B implies that B is a power of a Bessel process. Thus we reduce to the case when B is a BES(δ), that is a Bessel process of dimension δ started at $B_0 = 0$, with $0 < \delta < 2$ due to the recurrence assumption. Let P_α denote the probability measure or expectation operator governing B as a Bessel process of dimension $\delta = 2(1-\alpha)$, and $B^{\rm br}$ as a Bessel bridge of the same dimension. Let the speed measure be $m(dx) = x^{1-2\alpha}dx$. Now fix α, and compare (26) and (45) to see that the function $\mu_{\alpha,\lambda}(x)$ associated with the functional $F = M$ for the BES(δ) diffusion for $\delta = 2 - 2\alpha$ is determined by

$$[K\lambda^\alpha(1 + \mu_{\alpha,\lambda}(x))]^{-1} = g_{\alpha,\lambda}(0) - P_\alpha(e^{-\lambda T_x})g_{\alpha,\lambda}(x) \tag{46}$$

and the following well known evaluations. In terms of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$: (see e.g. [5, p. 115]):

$$g_{\alpha,\lambda}(x) = 2, (1-\alpha)^{-1}(\sqrt{\lambda/2})^{-\alpha} x^\alpha K_\alpha(\sqrt{2\lambda}x); \tag{47}$$

using $K_\alpha(y) \sim (1/2), (\alpha)(y/2)^{-\alpha}$ as $y \downarrow 0$ this implies

$$[K\lambda^\alpha]^{-1} = g_{\alpha,\lambda}(0) = , (\alpha), (1-\alpha)^{-1} 2^\alpha \lambda^{-\alpha}; \tag{48}$$

$$P_\alpha(e^{-\lambda T_x}) = \frac{(\sqrt{\lambda/2}x)^{-\alpha}}{, (1-\alpha) I_{-\alpha}(\sqrt{2\lambda}x)}. \tag{49}$$

Substitute these formulae in (46), use

$$2K_\alpha(y) = , (\alpha), (1-\alpha)(I_{-\alpha}(y) - I_\alpha(y)) \tag{50}$$

and simplify to obtain

$$\mu_{\alpha,\lambda}(x) = h_\alpha(\sqrt{2\lambda}x) - 1 \tag{51}$$

where $h_\alpha(x) := I_{-\alpha}(x)/I_\alpha(x)$.

Bessel Bridges For the ranked heights M_j^{br} of excursions of a standard Bessel bridge we deduce from (29) that for $0 < \alpha < 1$ corresponding to $0 < \delta < 2$ the P_α distribution of M_j^{br} is determined by the formula

$$P_\alpha(\sqrt{2\Gamma_\alpha} M_j^{\mathrm{br}} \geq x) = (1 - h_{-\alpha}(x))^j \qquad (x > 0,\ j = 1, 2, \dots) \tag{52}$$

where it is assumed that under P_α the random variable Γ_α has gamma(α) distribution and is independent of M_j^{br}. The previous formula (37) is the special case of this result with $\alpha = \frac{1}{2}, \delta = 1$. For $j = 1$ formula (52) determines the distribution of the maximum of a standard BES(δ) bridge for $0 < \delta < 2$. Kiefer [19] found an explicit formula for the distribution of the maximum of a standard BES(δ) bridge for all positive integer dimensions δ. See [6] for an alternative approach to formula (52), and [34] for further developments.

5 Evaluations at time τ_1

Recall that $(\tau_\ell, \ell \geq 0)$ is the inverse of the local time process at 0 for the self-similar Markov process B, with the local time process normalized so that (14) holds for some constants $\alpha \in (0, 1)$ and $K \in (0, \infty)$. Thus the Lévy measure of (τ_ℓ) is $K\Gamma(1-\alpha)^{-1}\Lambda_\alpha$, where for $\theta > 0$ we let Λ_θ denote the σ-finite measure on $(0, \infty)$ with

$$\Lambda_\theta(x, \infty) = x^{-\theta} \qquad (x > 0). \tag{53}$$

Proposition 10 *Let F be a γ-homogeneous functional of excursions of B. For $\ell > 0$ let $\mathcal{F}_\ell$ denote the random set of values of $F(e)$ obtained as e ranges over the countable collection of excursions of B away from 0 which are completed by time τ_ℓ. Then $\mathcal{F}_\ell$ is the set of points of a PPP with intensity*

$$\ell C_F \Lambda_{\alpha/\gamma} \text{ where } C_F = \frac{K}{\Gamma(1-\alpha)} E(F_*^{\alpha/\gamma}) \tag{54}$$

with $F_* := F(B^{\mathrm{ex}})$ for a standard B-excursion B^{ex}. If $E(F_*^{\alpha/\gamma}) = \infty$ then $\mathcal{F}_\ell$ is almost surely dense in $(0, \infty)$, whereas if $E(F_*^{\alpha/\gamma}) < \infty$ then $\mathcal{F}_\ell = \{F_j, j = 1, 2, \dots\}$ where $F_1 > F_2 > \dots > 0$ almost surely, with

$$F_j = (\ell C_F)^{\gamma/\alpha} \Gamma_j^{-\gamma/\alpha} \qquad (j = 1, 2, \dots) \tag{55}$$

where $\Gamma_j = \sum_{i=1}^{j} \varepsilon_i$ for i.i.d. standard exponential variables ε_i.

Remark. The definition of , $_j$ and ε_i involves both ℓ and F, but this dependence is suppressed in the notation as we regard both ℓ and F as fixed.

Proof. The Poisson character of $\mathcal{F}_\ell$ is an immediate consequence of Itô's description of the PPP of excursions of B up to time τ_ℓ. The intensity measure is computed as follows. First, for $F = V$, the lifetime of an excursion, with $\gamma = 1$, the random set $\mathcal{F}_\ell$ is the collection of jumps of the process $(\tau_u, 0 \leq u \leq \ell)$, say $\{V_j\}$ where V_j is the jth longest duration of an excursion of B up to time τ_ℓ. It is well known that the V_j are the ranked points of a PPP with intensity measure ℓK, $(1-\alpha)^{-1}\Lambda_\alpha$, which for $\ell = 1$ is also the Lévy measure of the subordinator $(\tau_u, u \geq 0)$. Next, for a general γ-homogeneous functional F let $\widetilde{F}_j$ denote the value of $F(e)$ for the excursion e of B whose length is V_j. Let B_j^{ex} denote the standard excursion obtained by rescaling the excursion of length V_j to have length 1. Then

$$\widetilde{F}_j = V_j^\gamma F(B_j^{\mathrm{ex}}) \tag{56}$$

where, due to the Poisson character of the Itô excursion process, and the self-similarity of B, the B_j^{ex} are independent copies of B^{ex}, and the sequence (B_j^{ex}) is independent of the sequence (V_j). It follows easily that the intensity measure of $\mathcal{F}_\ell$ is as specified in (54). The remaining assertions are implied by standard properties of Poisson processes.

Example. Let B be BM and let $F(e) := M(e)$ be the absolute maximum of excursion e. With the usual normalization of local time as occupation density relative to Lebesgue measure, $K = \sqrt{2}$ and the lengths V_j form a PPP$(\sqrt{2/\pi}\Lambda_{1/2})$. It is well known [35, p. 485] that the rate of excursions whose absolute value exceeds x is x^{-1}. So $C_M = 1$, and the ranked absolute maxima M_j of excursions up to time τ_1 are the points of a PPP(Λ_1). By application of the previous discussion to this case with $\alpha = \gamma = 1/2$ we deduce the formula of [4, p. 72 (2.f)] for the expected value of M_*, the absolute maximum of a standard Brownian excursion: $E(M_*) = \sqrt{\pi/2}$.

6 Dual formulae

The main purpose of this section is to give an alternative expression for the basic function $\mu_\lambda(x)$ in terms of the dual description of the excursion process obtained by conditioning on F-values rather than on lengths. The formulae of this section generalize results of Biane-Yor [4, §(3.3)] for F the maximum functional of a Brownian or Bessel excursions.

In the setting of Proposition 10, take $\ell = 1$, and assume that the distribution of $F_* := F(B^{\mathrm{ex}})$ has a density, say

$$f(x) := P(F_* \in dx)/dx \qquad (x > 0).$$

From (56), the $(V_j, \widetilde{F}_j)$ are the points of a PPP on $(0,\infty)^2$ with intensity measure $\lambda(t,x)\,dt\,dx$ where for $t, x > 0$

$$\lambda(t,x) := K_\alpha \alpha\, t^{-\alpha-1} t^{-\gamma} f(x/t^\gamma) \text{ with } K_\alpha := \frac{K}{\Gamma(1-\alpha)}. \tag{57}$$

Integrate out t and make the change of variable $x/t^\gamma = w$ to confirm the earlier claim that the $\widetilde{F}_j$ are the points of a PPP whose intensity measure has density at $x > 0$ equal to

$$\int_0^\infty K_\alpha \alpha t^{-\alpha-\gamma-1} f(x/t^\gamma) dt = K_\alpha E(F_*^{\alpha/\gamma})\, \tfrac{\alpha}{\gamma}\, x^{-\frac{\alpha}{\gamma}-1} \tag{58}$$

Assuming now that $E(F_*^{\alpha/\gamma}) < \infty$, take $x = 1$ in (58) to see that the formula

$$g(t) := (E(F_*^{\alpha/\gamma}))^{-1}\, \gamma\, t^{-\alpha-\gamma-1} f(1/t^\gamma) \tag{59}$$

defines a probability density on $(0,\infty)$. Now (57) can be recast as

$$\lambda(t,x) = K_\alpha E(F_*^{\alpha/\gamma})\, \tfrac{\alpha}{\gamma}\, x^{-\frac{\alpha}{\gamma}-1} x^{-\frac{1}{\gamma}}\, g(t/x^{\frac{1}{\gamma}}). \tag{60}$$

It follows that if $\widehat{V}_j$ denotes the length of the excursion of B up to time τ_1 whose F-value is F_j, the jth largest value of $F(e)$ as e ranges over all excursions of B completed by time τ_1, then

$$\widehat{V}_j = F_j^{1/\gamma}\, \bar{V}_j \tag{61}$$

where the $\bar{V}_j$ are independent random variables with common density g, independent also of the F_j.

In terms of Itô's law of excursions $n(de)$ defined on the space of excursion paths with $e := (e_t, 0 \le t \le V(e))$, the density $\lambda(t,x)$ is the joint density of the n-distribution of $(V(e), F(e))$. The density $x \to t^{-\gamma} f(x/t^\gamma)$ then serves as the n-conditional density of $F(e)$ given $V(e) = t$, and the density $t \to x^{-\frac{1}{\gamma}}\, g(t/x^{\frac{1}{\gamma}})$ serves as the n-conditional density of $V(e)$ given $F(e) = x$. Put another way, f is the probability density of F_*, the F-value of an excursion of B that is either conditioned or scaled to have lifetime 1, while g is the probability density of the lifetime $\bar{V}$ of an excursion of B that is conditioned or scaled to have its F-value equal to 1.

Note from (59) that

$$P(\bar{V} \in dt) = (E(F_*^{\alpha/\gamma}))^{-1}\, t^{-\alpha} P(F_*^{-1/\gamma} \in dt) \tag{62}$$

That is to say, for every non-negative Borel measurable function h

$$E[h(\bar{V})] = (E(F_*^{\alpha/\gamma}))^{-1} E[F_*^{\alpha/\gamma} h(F_*^{-1/\gamma})] \tag{63}$$

In particular, for $h(x) = x^\alpha$, there is the identity

$$E(\bar{V}^\alpha) = (E(F_*^{\alpha/\gamma}))^{-1} \tag{64}$$

Alternatively, we can take $x = 1/t^\gamma$ in (59) to express f in terms of g as

$$f(x) = E(F_*^{\alpha/\gamma})\,\gamma^{-1} x^{-(\alpha+\gamma+1)/\gamma} g(x^{-1/\gamma}) \tag{65}$$

Take $h(x) = k(x^{-\gamma})x^\alpha$ in (63) and use (64) to see that for every non-negative Borel measurable function k there is the identity

$$E[k(F_*)] = (E(\bar{V}^\alpha))^{-1} E[\bar{V}^\alpha k(\bar{V}^{-\gamma})] \tag{66}$$

Returning to consideration of the function $\mu_\lambda(x)$ as in Theorem 3, we now deduce the following corollary from that theorem:

Corollary 11 *Let χ denote the Laplace transform of the lifetime of an excursion conditioned to have F-value 1, that is for $\bar{V}$ with density g as in* (59)

$$\chi(\lambda) := E\exp(-\lambda\bar{V}) = \int_0^\infty g(t)e^{-\lambda t}dt \tag{67}$$

and let

$$\phi(\lambda) := \int_1^\infty \frac{\alpha}{\gamma}\, y^{-\frac{\alpha}{\gamma}-1}\chi(y^{1/\gamma}\lambda)\,dy = \alpha\,\lambda^\alpha \int_\lambda^\infty \frac{\chi(u)}{u^{\alpha+1}}\,du \tag{68}$$

which is the Laplace transform of the lifetime of the first excursion whose F-value exceeds 1. Then

$$\mu_\lambda(x) = \lambda^{-\alpha},\,(1-\alpha)^{-1}E(F_*^{\alpha/\gamma})\,x^{-\alpha/\gamma}\phi(x^{1/\gamma}\lambda). \tag{69}$$

Proof. Recall from the proof of Theorem 3 that $K\lambda^\alpha\mu_\lambda(x)$ is the rate of excursions e with $F(e) > x$, counting only excursions that survive killing at rate λ. This rate is the product of two factors, the rate of excursions e such that $F(e) > x$, and the probability that $e_{(F>x)}$ survives killing at rate λ, where $e_{(F>x)}$ is the first excursion e such that $F(e) > x$. By formula (54), the first factor equals K, $(1-\alpha)^{-1}E(F_*^{\alpha/\gamma})x^{-\alpha/\gamma}$, while the second factor is found by conditioning on $F(e_{(F>x)})$ to be

$$E(\exp(-\lambda V(e_{(F>x)}))) = \phi(x^{1/\gamma}\lambda) \tag{70}$$

and (69) results. □

Example. Suppose that B is a BM and let $F(e) := M(e)$ be the absolute maximum of excursion e. Now $\alpha = \gamma = 1/2$. According to Williams' decomposition of Brownian excursions at their maximum [44], under Itô's law n given $M = x$ the excursion decomposes into two independent BES(3) fragments, each stopped at its first hitting time of x. So g in this example is the convolution $g = g_1 * g_1$ where g_1 is the density of the hitting time of 1 by BES(3), and f is the density of M_*, the maximum of the standard Brownian excursion, so f is determined by differentiation of Chung's formula (8). According to (65) in this case, these two probability densities f and g are related by

$$f(m) = \sqrt{2\pi} m^{-4} g(m^{-2}). \tag{71}$$

Equivalently, from (66), if $T = T_1 + T_1'$ where T_1 and T_1' are the hitting times of 1 by two independent BES(3) processes starting from 0, then for every non-negative Borel function k

$$E[k(M_*)] = \sqrt{\pi/2}\, E[k(T^{-1/2})T^{1/2}]. \tag{72}$$

This is the basic *agreement formula* of Biane-Yor [4, (2.f)], who also gave the instance of (66) which determines the distribution of the maximum of the standard excursion of a recurrent Bessel process B. The explicit forms of the functions χ and ϕ in this example are recorded later in formula (87). See [45, 29, 3] for further developments. For other homogeneous functionals F of a Brownian excursion besides $F = M$, for example the area of an excursion, it is not evident how to provide such an explicit description of the law of the excursion given $F = 1$. Still, formula (66) describes explicitly how the distribution of the lifetime $\bar{V}$ of such a conditioned excursion is related to the distribution of F for a standard excursion of length 1.

7 The joint law of τ_1 and $F_j(\tau_1)$

As a consequence of (61), the stable(α) random variable τ_1 has been represented as

$$\tau_1 = \sum_{j=1}^{\infty} F_j^{\eta}\, \bar{V}_j \qquad \text{for } \eta := 1/\gamma \tag{73}$$

where the $\bar{V}_j$ are independent copies of the lifetime of a B-excursion conditioned to have F-value equal to 1, and the F_j are the ranked points of a PPP $(c\Lambda_\theta)$ that is

$$F_j = c^{1/\theta},{}_j^{-1/\theta} \qquad (j = 1, 2, \ldots) \tag{74}$$

where $, _j = \sum_{i=1}^{j} \varepsilon_i$ for i.i.d. standard exponential variables ε_i. Here $\theta = \alpha/\gamma$, and

$$c = C_F := K, (1-\alpha)^{-1}(E\bar{V}_1^{\alpha})^{-1} = K, (1-\alpha)^{-1}E(F_*^{\eta\alpha}).$$

Let us continue with the following slightly more general assumptions.

Assumptions. *Let τ_1 be defined by (73) for $\bar{V}_j$ independent copies of an arbitrary positive random variable $\bar{V}_1$ with $E(\bar{V}_1^{\alpha}) < \infty$, and F_j constructed independently of the $\bar{V}_j$ as in (74), for arbitrary*

$$\eta > 0, c > 0 \text{ and } \theta = \eta\alpha \text{ for } 0 < \alpha < 1.$$

Then the $(, _j, \bar{V}_j)$ are the points of a PPP on $(0,\infty)^2$ with intensity $dx\, P(\bar{V}_1 \in dy)$, hence the random measure $\sum_j 1(F_j^{\eta}\bar{V}_j \in \cdot\,)$ on $(0,\infty)$ is Poisson with intensity measure $cE(\bar{V}_1^{\alpha})\Lambda_{\alpha}(\,\cdot\,)$, and the distribution of τ_1 is stable with index α, with Laplace transform

$$E\exp(-\lambda\tau_1) = \exp(-c, (1-\alpha)E(\bar{V}_1^{\alpha})\lambda^{\alpha}) \tag{75}$$

This is a particular case of a well known construction of stable variables [38]. The following proposition records some features of the joint distribution of $F_1, \ldots, F_n$ involved in the representation (73) of a stable(α) variable τ_1.

Proposition 12 *With the above assumptions, let*

$$\chi(\lambda) := E\exp(-\lambda\bar{V}_j) \qquad\qquad (\lambda \geq 0) \tag{76}$$

and let functions ϕ and ψ be defined as follows:

$$\phi(\lambda) := \lambda^{\alpha}\int_{\lambda}^{\infty}\frac{\alpha\chi(u)}{u^{\alpha+1}}\,du; \quad \psi(\lambda) := \phi(\lambda) + , (1-\alpha)E(\bar{V}_1^{\alpha})\lambda^{\alpha}. \tag{77}$$

Then for each $n = 1, 2, \ldots$ the following formulae hold:

$$E\left[\exp(-\lambda\tau_1/F_n^{\eta}) \,|\, F_n\right] = (\phi(\lambda))^{n-1}\chi(\lambda)\exp\left[-cF_n^{-\theta}(\psi(\lambda)-1)\right]; \tag{78}$$

$$E\exp(-\lambda\tau_1/F_n^{\eta}) = (\phi(\lambda))^{n-1}\chi(\lambda)(\psi(\lambda))^{-n}; \tag{79}$$

$$E\left[\exp(-\lambda\tau_1) \,|\, F_1, \ldots, F_n\right] = \left(\prod_{j=1}^{n}\chi(F_j^{\eta}\lambda)\right)\exp\left(-cF_n^{-\theta}(\psi(F_n^{\eta}\lambda)-1)\right). \tag{80}$$

Proof. Let $\Delta_j := F_j^\eta$, so the Δ_j are the points of a PPP with intensity $c\Lambda_\alpha$, and let $\Gamma_j = c\Delta_j^{-\alpha} = cF_j^{-\theta}$ be the gamma variables defined implicitly by (74). Fix n and rewrite the definition of τ_1 as

$$\frac{\tau_1}{\Delta_n} = \sum_{j=1}^{n-1} \frac{\Delta_j}{\Delta_n} \bar{V}_j \; + \; \bar{V}_n \; + \; \sum_{j=n+1}^{\infty} \frac{\Delta_j}{\Delta_n} \bar{V}_j \tag{81}$$

and then apply [30, Lemma 24]. According to that lemma, conditionally given Δ_n the Δ_j/Δ_n for $1 \le j \le n-1$ are distributed like the order statistics of $n-1$ independent random variables $W_1, \ldots, W_{n-1}$ whose common distribution is the restriction of $\Lambda_\alpha(\cdot)$ to $(1, \infty)$, while the Δ_j/Δ_n for $n < j < \infty$ are distributed like the ranked points of a PPP whose intensity is the restriction to $(0,1)$ of $\Gamma_n \Lambda_\alpha(\cdot)$, where $\Gamma_n = c\Delta_n^{-\alpha}$. Moreover, the random vectors $(\Delta_j/\Delta_n, 1 \le j \le n-1)$ and $(\Delta_j/\Delta_n, n < j < \infty)$ are independent. From this description we obtain

$$E \exp\left(-\lambda \sum_{j=1}^{n-1} \frac{\Delta_j}{\Delta_n} \bar{V}_j \right) = \left(\int_1^\infty \frac{\alpha}{x^{\alpha+1}} \chi(\lambda x)\, dx \right)^{n-1} = (\phi(\lambda))^{n-1} \tag{82}$$

and, as will be verified below,

$$E \exp\left(-\lambda \sum_{j>n} \frac{\Delta_j}{\Delta_n} \bar{V}_j \,\middle|\, c\Delta_n^{-\alpha} = x \right) = \exp\left[-x(\psi(\lambda) - 1) \right]. \tag{83}$$

Formula (78) follows from (82), (83) and the decomposition (81) of τ_1/Δ_n into independent components. Formula (79) now follows from (78) by integration with respect to the distribution of $\Gamma_n = c\Delta_n^{-\alpha}$, and (80) can now be read directly from (81). To complete the argument, it only remains to check (83). The previous discussion yields (83) with $v(\lambda)$ instead of $\psi(\lambda) - 1$, for the Laplace exponent $v(\lambda)$ defined as follows:

$$E \exp(-\lambda \Sigma_{(0,1]}) =: \exp(-v(\lambda)) \tag{84}$$

where for a subinterval B of $[0, \infty)$ we set

$$\Sigma_B := \sum_{j=1}^{\infty} \bar{\Delta}_j 1(\bar{\Delta}_j \in B) \bar{V}_j \tag{85}$$

for $\bar{\Delta}_j$ the points of a PPP with intensity Λ_α independent of the $\bar{V}_j$. But $\Sigma_{(0,1]}$ and $\Sigma_{(1,\infty)}$ are independent with sum $\Sigma_{[0,\infty)}$ which has a stable distribution with index α. By conditioning on the Poisson(1) distributed number of j such that $\bar{\Delta}_j \geq 1$ and applying (82) we can compute

$$E\exp(-\lambda\Sigma_{[1,\infty)}) = \sum_{m=0}^{\infty} \frac{e^{-1}}{m!}\phi(\lambda)^m = \exp(\phi(\lambda) - 1)$$

and hence, reading the Laplace transform of $\Sigma_{[0,\infty)}$ from (75) with $c = 1$,

$$E\exp(-\lambda\Sigma_{(0,1]}) = \frac{E\exp(-\lambda\Sigma_{[0,\infty)})}{E\exp(-\lambda\Sigma_{[1,\infty)})} = \frac{\exp(-\,,\,(1-\alpha)E(\bar{V}_1^{\alpha})\lambda^{\alpha})}{\exp(\phi(\lambda) - 1)} = \exp(1 - \psi(\lambda))$$

as claimed. □

Remarks.

(a) The proof shows that both $\phi(\lambda)$ and $1/\psi(\lambda)$ are the Laplace transforms of probability distributions on $(0, \infty)$. In particular, we deduce from (82) that

$$\phi(\lambda) = E\exp(-\lambda U^{-1/\alpha}\bar{V}_1) \tag{86}$$

where $U := (\Delta_1/\Delta_2)^{-\alpha} = \,,\,_1/\,,\,_2$ has uniform distribution on $(0, 1)$, and is independent of $\bar{V}_1$. So formula (79) shows that τ_1/Δ_n is distributed as the sum of $2n$ independent random variables, with $n - 1$ variables distributed like $U^{-1/\alpha}\bar{V}_1$, one distributed like $\bar{V}_1$, and n distributed like Σ_1 with $E\exp(-\lambda\Sigma_1) = 1/\psi(\lambda)$. Equation (83) shows that such a Σ_1 may be constructed as $\Sigma_1 := \sum_{j>1} \frac{\Delta_j}{\Delta_1}\bar{V}_j$.

(b) When the distribution of $\bar{V}_j$ is degenerate at 1, the functions χ, ϕ and ψ reduce to

$$\chi(\lambda) = e^{-\lambda}; \quad \phi(\lambda) = \int_1^{\infty} e^{-\lambda x}\alpha x^{-\alpha-1}dx; \quad \psi(\lambda) = \,,\,(1-\alpha)\lambda^{\alpha} + \phi(\lambda).$$

The results of the proposition in this case all appear in [30, §2 and §4], with the notations $\phi_\alpha(\lambda)$ and $\psi_\alpha(\lambda)$ instead of $\phi(\lambda)$ and $\psi(\lambda)$. If τ_1 is the value at time 1 of a subordinator $(\tau_\ell, \ell \geq 0)$, a random variable X whose Laplace transform is $1/\psi(\lambda)$ can be constructed as $W := \tau_{S-}$ where S is the least ℓ such that $\tau_\ell - \tau_{\ell-} > 1$. See [30] for various further developments in this case, and an explanation of why the same distributions appear in the work of Darling [11], Lamperti [22] and Wendel [42].

(c) With τ_1 the value at time 1 of the inverse local time at level 0 for a B a BM started at 0, let $\Delta_n := M_n^2$ be the squared height of the nth highest excursion of $|B|$ up to time τ_1, and let $\bar{V}_n$ be the sum of two independent hitting times of 1 by a BES(3) process. The functions χ, ϕ and ψ are then determined by

$$\chi(\tfrac{1}{2}\nu^2) = \left(\frac{\nu}{\sinh\nu}\right)^2; \quad \phi(\tfrac{1}{2}\nu^2) = \left(\frac{\nu}{\sinh\nu}\right) e^{-\nu}; \quad \psi(\tfrac{1}{2}\nu^2) = \frac{\nu}{\tanh\nu}. \tag{87}$$

Formula (79) in this instance simplifies to

$$E\left[\exp\left(-\tfrac{1}{2}\nu^2\frac{\tau_1}{M_n^2}\right)\right] = \left(\frac{2\nu}{\sinh(2\nu)}\right)\left(\frac{2}{e^{2\nu}+1}\right)^{n-1}. \tag{88}$$

For $n = 1$ this identity is due to Knight [21]. See also [31] for a number of other extensions of Knight's identity. As shown by Knight [20], random variables with Laplace transforms ϕ and $1/\psi$ can be constructed in this case as follows. Let T_1 be the first hitting time of 1 by $|B|$, and let (G_1, D_1) be the excursion interval of B straddling time T_1. Then ϕ and $1/\psi$ are the Laplace transforms of $D_1 - G_1$ and G_1 respectively. Similar interpretations of ϕ and $1/\psi$ can be given for a more general F, as indicated already in Corollary 11 in the case of ϕ. See also [33] for a study of similar Laplace transforms related to excursions of a one-dimensional diffusion.

(e) It is immediately apparent from the definition (77) of ϕ and ψ that these two functions are solutions of the differential equation

$$\lambda f'(\lambda) - \alpha f(\lambda) = -\alpha\chi(\lambda) \tag{89}$$

and that there is the Wronskian formula

$$\phi(\lambda)\psi'(\lambda) - \phi'(\lambda)\psi(\lambda) = \alpha K\lambda^{\alpha-1}\chi(\lambda). \tag{90}$$

These remarks extend formulae (76) and (87) of [30]. Note in particular from (89) that χ is readily recovered from either ϕ or ψ.

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