Two recursive decompositions of Brownian bridge related to the asymptotics of random mappings *

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Abstract

Aldous and Pitman (1994) studied asymptotic distributions as $n \to \infty$, of various functionals of a uniform random mapping of the set $\{1, \ldots, n\}$, by constructing a *mapping-walk* and showing these random walks converge weakly to a reflecting Brownian bridge. Two different ways to encode a mapping as a walk lead to two different decompositions of the Brownian bridge, each defined by cutting the path of the bridge at an increasing sequence of recursively defined random times in the zero set of the bridge. The random mapping asymptotics entail some remarkable identities involving the random occupation measures of the bridge fragments defined by these decompositions. We derive various extensions of these identities for Brownian and Bessel bridges, and characterize the distributions of various path fragments

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involved, using the Lévy-Itô theory of Poisson processes of excursions for a self-similar Markov process whose zero set is the range of a stable subordinator of index $\alpha \in (0,1)$.

Keywords Brownian bridge, Brownian excursion, local time, occupation measure, stable subordinator, self-similar Markov process, Bessel process, path decomposition, Poisson-Dirichlet distribution, pseudo-bridge, random mapping, size-biased sampling, weak convergence, exchangeable interval partition.

Contents

1 Introduction

In a previous paper [2] we showed how features of a uniformly distributed random mapping M_n, from $[n] := \{1, 2, \ldots, n\}$ to itself, could be encoded as functionals of a particular non-Markovian random walk on the non-negative integers. This *mapping-walk*, suitably rescaled, converges weakly in $C[0,1]$ as $n \to \infty$ to the distribution of the reflecting Brownian bridge defined by the absolute value of a *standard Brownian bridge* B^{br} with $B_0^{\text{br}} = B_1^{\text{br}} = 0$ obtained by conditioning a standard Brownian motion B on $B_1 = 0$. Two important features of a mapping are the vector of sizes of connected components of its digraph, and the vector of sizes of cycles in its digraph. Results of [2] imply that for a uniform random mapping, as $n \to \infty$, the component sizes rescaled by n, jointly with corresponding cycle sizes rescaled by $\sqrt{n}$, converge in distribution to a limiting bivariate sequence of random variables $(\lambda_{I_j}, L^0_{I_j})_{j=1,2,\ldots}$ where $(I_j)_{j=1,2,\ldots}$ is a random interval partition of $[0,1]$, with λ_{I_j} the length of I_j and $L^0_{I_j}$ the increment of local time of B^{br} at 0 over the interval I_j. With the convention for ordering connected components of the mapping digraph used in [2], the limiting interval partition is $(I_j) = (I_j^D)$, according to the following definition. Here, and throughout the paper, $U, U_1, U_2, \ldots$ denotes a sequence of independent uniform $(0,1)$ variables, independent of B^{br}.

Definition 1 (The D-partition [2]) Let $I_j^D := [D_{V_{j-1}}, D_{V_j}]$ where $V_0 = D_{V_0} = 0$ and V_j is defined inductively along with the D_{V_j} for $j \geq 1$ as follows: given that D_{V_i} and V_i have been defined for $0 \leq i < j$, let

$$V_j := D_{V_{j-1}} + U_j(1 - D_{V_{j-1}}),$$

so V_j is uniform on $[D_{V_{j-1}}, 1]$ given B^{br} and (V_i, D_{V_i}) for $0 \leq i < j$, and let

$$D_{V_j} := \inf\{t \geq V_j : B_t^{\text{br}} = 0\}.$$

On the other hand, a variation of the main result of [2] shows that with a different ordering convention, the mapping component sizes rescaled by n, jointly with their cycle sizes rescaled by $\sqrt{n}$, have a limit distribution specified by the sequence of lengths and Brownian local times $(\lambda_{I_j}, L^0_{I_j})_{j=1,2,\ldots}$ a differently defined limiting interval partition. This is the partition $(I_j) = (I_j^T)$ defined as follows using the local time process $(L_u^0, 0 \leq u \leq 1)$ of B^{br} at 0:

Definition 2 (The T-partition) Let $I_j^T := [T_{j-1}, T_j]$ where $T_0 := 0$, $\hat{V}_0 := 0$, and for $j \geq 1$

$$\hat{V}_j := 1 - \prod_{i=1}^{j}(1 - U_i), \tag{1}$$

so $\hat{V}_j$ is uniform on $[\hat{V}_{j-1}, 1]$ given B^{br} and $(\hat{V}_i, T_i)$ for $0 \leq i < j$, and

$$T_j := \inf\{u : L_u^0 / LB_1 > \hat{V}_j\}.$$

For each of these two random interval partitions (I_j) we are interested in the distribution of the bivariate sequence of lengths and local times $(\lambda_{I_j}, L_{I_j}^0)_{j=1,2,\ldots}$ and the distribution of the associated path fragments $B^{\text{br}}[I_j]$ and standardized fragments $B_*^{\text{br}}[I_j]$. Here for a process $X := (X_t, t \in J)$ parameterized by an interval J, and $I = [G_I, D_I]$ a subinterval of J with length $\lambda_I := D_I - G_I > 0$, we denote by $X[I]$ or $X[G_I, D_I]$ the *fragment of* X *on* I, that is the process

$$X[I]_u := X_{G_I + u} \qquad (0 \leq u \leq \lambda_I). \tag{2}$$

We denote by $X_*[I]$ or $X_*[G_I, D_I]$ the *standardized fragment of* X *on* I, defined by the *Brownian scaling operation*

$$X_*[I]_u := \frac{X[I]_{u\lambda_I}}{\sqrt{\lambda_I}} := \frac{X_{G_I + u\lambda_I}}{\sqrt{\lambda_I}} \qquad (\,0 \leq u \leq 1). \tag{3}$$

Figure 1 illustrates these definitions for a typical path of $X = B^{\text{br}}$. Note that the first interval I_1^D of the D-partition ends at the time D_{U_1} of the first zero of B^{br} after a uniform$(0,1)$-distributed time U_1, whereas the first interval I_1^T of the T-partition ends at the time T_1 when the local time of B^{br} at 0 has reached a uniform$(0,1)$-distributed fraction of its ultimate value. As illustrated in Figure 1, the associated fragments of B^{br} are qualitatively different: $B_*^{\text{br}}[I_1^D]$ ends with an excursion while $B_*^{\text{br}}[I_1^T]$ does not.

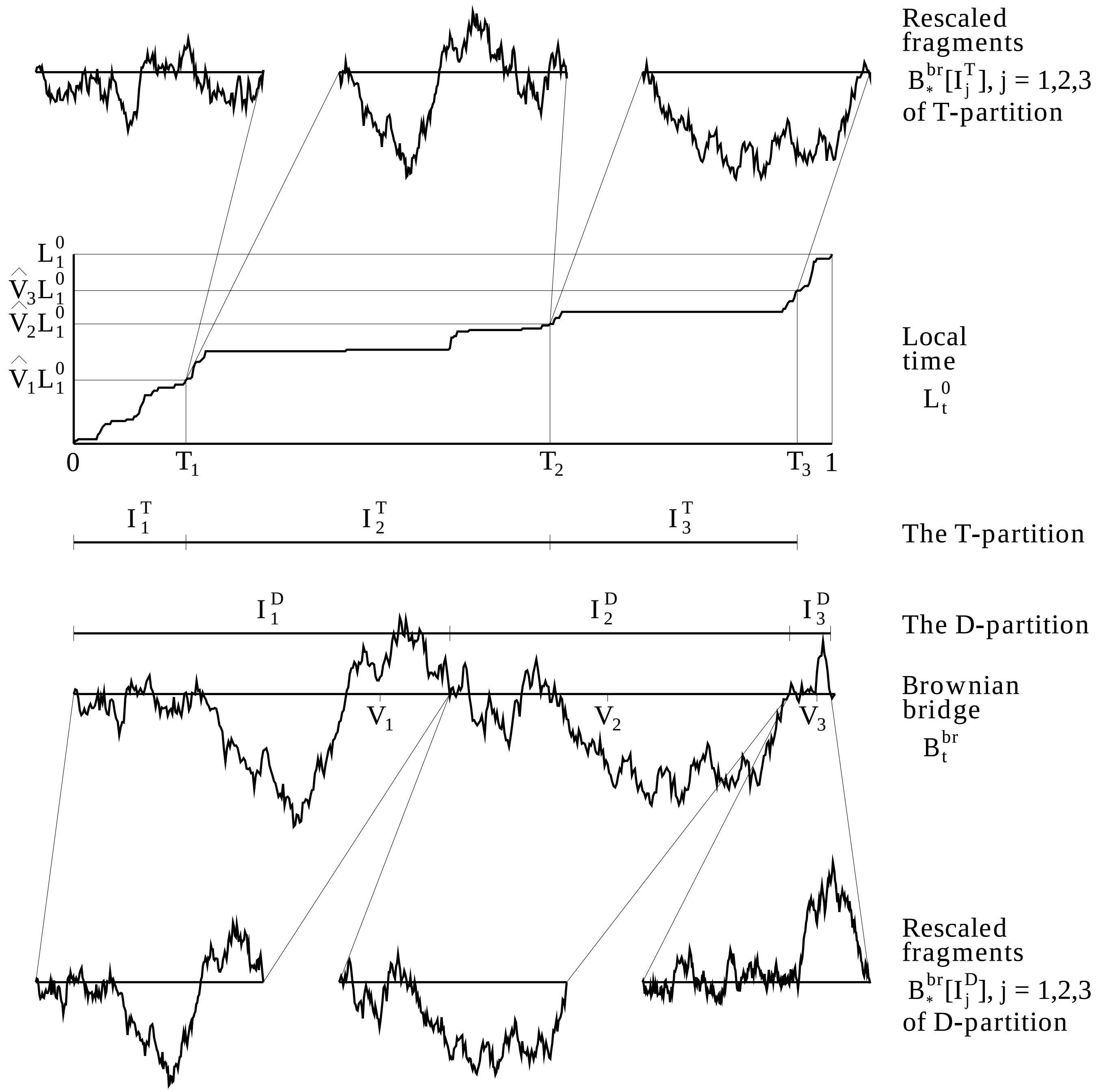

Figure 1. The two interval partitions.

Despite this difference between the fragments of B^{br} over the D- and T-partitions, the random mapping asymptotics have the following corollary. Let $(I^D_{(j)})$ and $(I^T_{(j)})$ denote the length-ranked D-partition and the length-ranked T-partition respectively, meaning $I^D_{(j)}$ is the jth longest interval in the D-partition, and $I^T_{(j)}$ is the jth longest interval in the T-partition.

Corollary 3 *Considering the four bivariate sequences $(\lambda_{I_j}, L^0_{I_j})_{j=1,2,\dots}$ of lengths and bridge local times at 0, for (I_j) one of the four random interval partitions of $[0,1]$ defined by (I^D_j), $(I^D_{(j)})$, (I^T_j) or $(I^T_{(j)})$,*
(i) *the bivariate sequence has the same distribution for $(I^D_{(j)})$ as for $(I^T_{(j)})$;*
(ii) *the bivariate sequence for (I^D_j) is the bivariate sequence for $(I^D_{(j)})$ in a length-biased order;*
(iii) *the bivariate sequence for (I^T_j) is the bivariate sequence for $(I^T_{(j)})$ in an L^0-biased order;*
(iv) *the sequence of local times $(L^0 I_j)$ has the same distribution for (I^D_j) as for (I^T_j), whereas the sequence of lengths (λ_{I_j}) does not.*

See [25, 26] for background about size-biased random orderings. To illustrate the meaning of (iii) for instance, for each $k \geq 1$, conditionally given the entire bivariate sequence $(\lambda_{I^T_{(j)}}, L^0_{I^T_{(j)}})_{j=1,2,\dots})$, the probability of the event $(I^T_1 = I^T_{(k)})$ is $L^0_{I^T_{(k)}}/L^0_1$, where $L^0_1 = \sum_j L^0_{I^T_{(j)}}$ almost surely. And given also $(I^T_1 = I^T_{(k)})$, for each $m \geq 1$ with $m \neq k$ the probability of the event $(I^T_2 = I^T_{(m)})$ is $L^0_{I^T_{(m)}}/(L^0_1 - L^0_{I^T_1})$ and so on. Put another way, parts (i)-(iii) of the corollary state that the bivariate sequence $(\lambda_{I_j}, L^0_{I_j})_{j=1,2,\dots}$ for $(I_j) = (I^D_j)$ is distributed like a length-biased rearrangement of the bivariate sequence for $(I_j) = (I^T_j)$, which is in turn distributed like an L^0-biased rearrangement of the bivariate sequence for $(I_j) = (I^D_j)$. Consequently, the distribution of any one of the four bivariate sequences determines the distribution of each of the others.

The rest of this paper is organized as follows. Section 2 explains how we discovered Corollary 3 by consideration of random mapping asymptotics. We recall the theorem from [2] which describes the asymptotics of mapping-walks in terms of the fragments of B^{br} defined by the D-partition, and present the companion result, for a different orderings of components, where the limit involves the fragments of B^{br} defined by the T-partition. Section 3 lays out our results regarding the decomposition of B^{br} into path fragments associated with the D- and T-partitions, in a way which does not depend the random mapping asymptotics. In particular, we describe the three different distributions of bivariate sequences featuring in the three parts of Corollary 3. We formulate and prove these results more generally, for B^{br} the standardized bridge of a recurrent self-similar Markov process B whose inverse local time process at 0 is a stable subordinator of index α for some $\alpha \in (0,1)$. So $\alpha = \frac{1}{2}$ for B a standard Brownian motion as supposed in previous paragraphs, and $\alpha = 1 - \delta/2$ for B a Bessel process of dimension $\delta \in (0,2)$. Some of the results in Section 3, like Corollary 3, can be viewed in the Brownian case as asymptotic counterparts (under weak convergence of mapping-walks) of some combinatorial symmetries of random mappings, discussed in Section 2.2. Other results in the Brownian case, especially those involving the method of Poissonization by random scaling [34, 35], are not obvious from the combinatorial perspective, but provide explicit limit distributions for functionals of uniform random mappings. See also [4] where we apply this method to characterize the asymptotic distribution of the diameter of the digraph of a uniform mapping. Sections 4 and 5 provide some proofs and further details of the main results in Section 3, while Section 6 contains various complements. In particular, we show in Section 6.2 that Corollary 3 holds even more generally for interval partitions (I_j^D) and (I_j^T) defined as before, but with the random zero set of B^{br} replaced by the complement of $\cup_j I_j^{\mathrm{ex}}$, where I_j^{ex} is any exchangeable random partition of $[0,1]$ into an infinite number of intervals, and $(L_u^0, 0 \le u \le 1)$ is the associated local time process, as defined by Kallenberg [16]. This is the limiting case of a corresponding result for a finite exchangeable interval partition of $[0,1]$, which we prove by a combinatorial argument.

In companion papers [3] and [1] we show that Brownian bridge asymptotics apply for models of random mappings more general than the uniform model, in particular for the *p-mapping* model [24, 29], and that proofs can be simplified by use of Joyal's bijection between mappings and trees. See also [30] for a recent review of the applications of Brownian motion and Poisson processes to the asymptotics of various kinds of large combinatorial objects, including partitions, trees, graphs, permutations, and mappings.

2 Random Mappings

In this section we explain how study of random mappings led us to consideration of the two interval partitions of Brownian bridge, and show how the distributions of path fragments of the bridge defined by these partitions encode various asymptotic distributions for mappings.

2.1 Mapping-walks and the two orderings

A mapping $M_n : [n] \to [n]$ can be identified with its digraph of edges $\{(i, M_n(i)),\ i \in [n]\}$. The connection between random mappings and Brownian bridge developed in [2] can be summarized as follows.

- A mapping digraph can be decomposed as a collection of *rooted trees* together with extra structure (*cycles, basins of attraction*).
- A rooted tree can be coded as a discrete *tree-walk*, a walk excursion starting and ending at 0.
- Given some ordering of tree-components, one can concatenate walk-excursions to define a discrete *mapping-walk* which codes M_n.
- For a uniform random mapping, the induced distribution on tree-components is such that the tree-walks, suitably normalized, converge to Brownian excursion as the tree size increases to infinity.
- So for a uniform random mapping, we expect the mapping-walks, suitably normalized, to converge to a limit process defined by some concatenation of Brownian excursions.
- With appropriate choice of ordering, the limit process is in fact reflecting Brownian bridge.

We now amplify this summary, emphasizing the only subtle issue – the choice of ordering. Fix a mapping M_n. It has a set of *cyclic points*

$$\mathcal{C}_n := \{i \in [n] : M_n^k(i) = i \text{ for some } k \geq 1\},$$

where M_n^k is the kth iterate of M_n. Let $\mathcal{T}_{n,c}$ be the set of vertices of the (perhaps trivial) tree component of the digraph with root $c \in \mathcal{C}_n$. The tree components are bundled by the disjoint cycles $\mathcal{C}_{n,j} \subseteq \mathcal{C}_n$ to form the *basins of attraction* (connected components) of the mapping digraph, say

$$\mathcal{B}_{n,j} := \bigcup_{c \in \mathcal{C}_{n,j}} \mathcal{T}_{n,c} \supseteq \mathcal{C}_{n,j} \text{ with } \bigcup_j \mathcal{B}_{n,j} = [n] \text{ and } \bigcup_j \mathcal{C}_{n,j} = \mathcal{C}_n \qquad (4)$$

where all three unions are disjoint unions, and the $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ are indexed in some way by $j = 1, \ldots, K_n$ say. The construction in [2] encodes the restriction of the digraph of M_n to each tree component $\mathcal{T}_{n,c}$ of size k by $2k$ steps of a *tree-walk* with increments ± 1 on the non-negative integers. The tree-walk proceeds by a suitable search of the set $\mathcal{T}_{n,c}$, making an excursion which starts at 0 and returns to 0 for the first time after $2k$ steps, after reaching a maximum level $1 + h_n(c)$, where $h_n(c)$ is the maximal height above c of all vertices of the tree $\mathcal{T}_{n,c}$ with root c, that is

$$h_n(c) = \max\{h : \exists i \in [n] \text{ with } M_n^h(i) = c \text{ and } M_n^j(i) \notin \mathcal{C}_n \text{ for } 0 \leq j < h\}. \tag{5}$$

It was shown in [5] that as $k \to \infty$, the distribution of the tree-walk for a k-vertex random tree, of the kind contained in the digraph of the uniform random mapping M_n for $k \leq n$, when scaled to have $2k$ steps of $\pm 1/\sqrt{k}$ per unit time, converges to the distribution $2B^{\text{ex}}$ for B^{ex} a standard Brownian excursion. Subsequent work [22] shows that the same result holds for a variety of codings of trees as walks. Consequently, any of these codings would serve our purpose in the following definitions.

We now define a mapping-walk (to code M_n) as a concatenation of its tree-walks, to make a walk of $2n$ steps starting and ending at 0 with exactly $|\mathcal{C}_n|$ returns to 0, one for each tree component of the mapping digraph. To retain useful information about M_n in the mapping-walk, we want the definition of the walk to respect the cycle and basin structure of the mapping. Here are two orderings that do so.

Definition 4 (Cycles-first ordering) Fix a mapping M_n from $[n]$ to $[n]$. If M_n has K_n cycles, first put the cycles in increasing order of their least elements, say $c_{n,1} < c_{n,2} < \ldots < c_{n,K_n}$. Let $\mathcal{C}_{n,j}$ be the cycle containing $c_{n,j}$, and let $\mathcal{B}_{n,j}$ be the basin containing $\mathcal{C}_{n,j}$. Within cycles, list the trees around the cycles, as follows. If the action of M_n takes $c_{n,j} \to c_{n,j,1} \to \cdots \to c_{n,j}$ for each $1 \leq j \leq K_n$, the tree components $\mathcal{T}_{n,c}$ are listed with c in the order

$$(\overbrace{c_{n,1,1}, \ldots, c_{n,1}}^{\mathcal{C}_{n,1}}, \overbrace{c_{n,2,1}, \ldots, c_{n,2}}^{\mathcal{C}_{n,2}}, \ldots\ldots, \overbrace{c_{n,K_n,1}, \ldots, c_{n,K_n}}^{\mathcal{C}_{n,K_n}}). \tag{6}$$

The *cycles-first mapping-walk* is obtained by concatenating the tree walks derived from M_n in this order. The *cycles-first search of* $[n]$ is the permutation $\sigma : [n] \to [n]$ where σ_j is the jth vertex of the digraph of M_n which is visited in the corresponding concatenation of tree searches.

Definition 5 (Basins-first ordering)[2] If M_n has K_n cycles, first put the basins $\mathcal{B}_{n,j}$ in increasing order of their least elements, say $1 = b_{n,1} < b_{n,2} < \ldots b_{n,K_n}$; let $c_{n,j} \in \mathcal{C}_{n,j}$ be the cyclic point at the root of the tree component containing $b_{n,j}$. Now list the trees around the cycles, just as in (6), but for the newly defined $c_{n,j}$ and $c_{n,j,i}$. Call the corresponding mapping-walk and search of $[n]$ the *basins-first mapping-walk* and *basins-first search.*

Be aware that the meaning of $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ now depends on the ordering convention. Rather than introduce two separate notations for the two orderings, we use the same notation for both, and indicate nearby which ordering is meant. Whichever ordering, the definitions of $\mathcal{B}_{n,j}$ and $\mathcal{C}_{n,j}$ are always linked by $\mathcal{B}_{n,j} \supseteq \mathcal{C}_{n,j}$, and (4) holds.

Let us briefly observe some similarities between the two mapping-walks. For each given basin B of M_n with say b elements, the restriction of M_n to B is encoded in a segment of each walk which equals at 0 at some time, and returns again to 0 after $2b$ more steps. If the basin contains exactly c cyclic points, this walk segment of $2b$ steps will be a concatenation of c excursions away from 0. Exactly where this segment of $2b$ steps appears in the mapping-walk depends on the ordering convention, as does the ordering of excursions away from 0 within the segment of $2b$ steps. However, many features of the action of M_n on the basin B are encoded in the same way in the two different stretches of length $2b$ in the two walks, despite the permutation of excursions. One example is the number of elements in the basin whose height above the cycles is h, which is encoded in either walk as the number of upcrossings from h to $h+1$ in the stretch of walk of length $2b$ corresponding to that basin.

2.2 Symmetry properties of random mappings

We now apply the definitions above to a uniform random mapping M_n. Of course, the random partition $\{\mathcal{B}_{n,j}\}_{j=1,\ldots,K_n}$ of $[n]$, and the random partition $\{\mathcal{C}_{n,j}\}_{j=1,\ldots,K_n}$ of $\mathcal{C}_n$, are the same no matter which ordering convention is used. Each random partition is *exchangeable*, meaning its distribution is invariant under the action of a permutation of $[n]$. Let us spell out some further symmetry properties, each of which turns out to have some analog in the limiting Brownian scheme.

(a) The cycles-first ordering has the following very strong symmetry property: conditionally given $|\mathcal{C}_n| = m$ the tree components in cycles-first ordering form an exchangeable sequence of m random subsets of $[n]$; moreover this exchangeable sequence is independent of the sequence of cycle sizes $|\mathcal{C}_{n,j}|$ with $\sum_j |\mathcal{C}_{n,j}| = m$. Consequently, given $|\mathcal{C}_n| = m$, the cycles-first mapping-walk is a concatenation of m exchangeable excursions away from 0, and this mapping-walk is independent of $|\mathcal{C}_{n,j}|, j = 1, 2, \ldots, K_n$.

(b) The basins-first ordering does not share the symmetry property above. But it has a different one: given that the basin $\mathcal{B}_{n,1}$ containing 1 has size $|\mathcal{B}_{n,1}| = b$, the action of M_n on $[n] - \mathcal{B}_{n,1}$ is that of a uniform random mapping of a set of $n - b$ elements. So given $|\mathcal{B}_{n,1}| = b$, the basins-first mapping-walk decomposes after $2b$ steps into two independent segments: the first $2b$ steps are distributed like the basins-first walk for a uniform mapping of $[b]$ conditioned to have a single basin, and the remaining $2(n-b)$ steps distributed like the basins-first walk associated with a uniform mapping of $[n-b]$.

(c) The sequence of basin sizes $(|\mathcal{B}_{n,j}|, 1 \le j \le K_n)$ does not have the same distribution for both orderings. For instance, if $|\mathcal{B}_{n,1}| = 1$ in the basins-first ordering then $|\mathcal{B}_{n,1}| = 1$ in the cycles-first ordering, but (for $n \ge 3$) not conversely. So the distribution of $|\mathcal{B}_{n,1}|$ must be different in the two orderings.

(d) For a given mapping M_n, the sequence of cycle sizes $(|\mathcal{C}_{n,j}|, 1 \le j \le K_n)$ may be different for the two different orderings. But for M_n with uniform distribution on $[n]^{[n]}$, the two sequences of cycle sizes have the same distribution: given $|\mathcal{C}_n| = m$, either sequence is distributed like the sizes of cycles of a uniform random permutation of $[m]$ in the (size-biased) order of least elements of the cycles. That is to say, given $|\mathcal{C}_n| = m$, the distribution of $|\mathcal{C}_{n,1}|$ is uniform on $[m]$; given $|\mathcal{C}_n| = m$ and $|\mathcal{C}_{n,1}|$ with $|\mathcal{C}_n| - |\mathcal{C}_{n,1}| = m_1$, the distribution of $|\mathcal{C}_{n,2}|$ is uniform on $[m_1]$, and so on. This is a well known property of uniform random permutations for the cycles-first ordering, and was shown for the basins-first ordering in [2, Lemma 22].

2.3 Brownian asymptotics for the mapping-walks

We now come to the main point of Section 2: the definitions of the interval partitions of Brownian bridge are motivated by the following theorem.

Theorem 6 *The scaled mapping-walk $(M_u^{[n]}, 0 \le u \le 1)$, with $2n$ steps of $\pm 1/\sqrt{n}$ per unit time, for either the cycles-first or the basins-first ordering of excursions corresponding to tree components, converges in distribution to*

$2|B^{\mathrm{br}}|$ jointly with

$$\frac{|\mathcal{C}_n|}{\sqrt{n}} \xrightarrow{d} L_1^0 \tag{7}$$

where $(L_u^0, 0 \leq u \leq 1)$ is the process of local time at 0 of B^{br}, normalized so that $P(L_1^0 > \ell) = e^{-\frac{1}{2}\ell^2}$. Moreover,

(i) *for the cycles-first ordering, with the cycles $\mathcal{B}_{n,j}$ in order of their least elements, these two limits in distribution hold jointly with*

$$\left(\frac{|\mathcal{B}_{n,j}|}{n}, \frac{|\mathcal{C}_{n,j}|}{\sqrt{n}}\right) \xrightarrow{d} (\lambda_{I_j}, L_{I_j}^0) \tag{8}$$

as j varies, where the limits are the lengths and increments of local time of B^{br} at 0 associated with the interval partition $(I_j) := (I_j^T)$; whereas

(ii) [2] *for the basins-first ordering, with the basins $\mathcal{B}_{n,j}$ listed in order of their least elements, the same is true, provided the limiting interval partition is defined instead by $(I_j) := (I_j^D)$.*

The result for basins-first ordering is part of [2, Theorem 8]. The variant for cycles-first ordering can be established by a variation of the argument in [2], exploiting the exchangeability property of the cycles-first ordering (Section 2.2(a) instead of Section 2.2 (b)). See also [10] and [1] for alternate approaches to the basic result of [2].

We now explain how we first discovered the facts about Brownian bridge presented in Corollary 3 by consideration Theorem 6 and the symmetry properties of Section 2.2. Note however that we show in Section 6 that the results of Corollary 3 hold much more generally, so these results do not really involve much of the rich combinatorial structure of mapping digraphs involved in Theorem 6.

(a) In the basins-first ordering, the first basin is by definition the basin containing element 1, and its walk-segment ends at the first time that the walk returns to 0 after the basins-first search has reached element 1. By exchangeability one can replace element 1 by a uniform random element, so the walk-segment corresponds asymptotically to the walk-segment ending at the first time of reaching 0 after a uniform random time on $[0, 2n]$. Rescaling, this corresponds to the time interval $[0, D_{V_1}]$ in Definition 1.

Now consider the cycles-first ordering. The first basin is by definition the basin containing the smallest-numbered cyclic element $c_{n,1}$, and its walk-segment ends at the first time after reaching element $c_{n,1}$ that the walk returns

to 0. By exchangeability one can replace element $c_{n,1}$ by a uniform random *cyclic* element, so the walk-segment corresponds asymptotically to the walk-segment ending at the first time of reaching 0 after visiting $U^*|\mathcal{C}_n|$ cyclic vertices, where U^* has uniform$[0,1]$ distribution. Rescaling, this corresponds to the time interval $[0,T_1]$ in Definition 2.
(b) The recursive property of the basins-first ordering in Section 2.2(b) plainly corresponds, under the asymptotics of Theorem 6, to the recursive decomposition of Brownian bridge at time D_{V_1} described later in Lemma 8.
(c) In Section 2.2(c) we observed that the distribution of $\mathcal{B}_{n,1}$ was different in the two orderings. This difference persists in the limit: Theorem 6 and the calculation below (26) imply

$$\lim_n n^{-1} E|\mathcal{B}_{n,1}| = \begin{cases} E(D_{V_1}) = 2/3 & \text{(for the basins-first ordering)} \\ E(T_1) = 1/2 & \text{(for the cycles-first ordering).} \end{cases}$$

(d) It is well known [41] that the asymptotic distribution as $n \to \infty$ of the fractions of elements in cycles of a random permutation of $[n]$, with the cycles in order of their least elements, (which amounts to a size-biased random order by exchangeability), is the *uniform stick-breaking sequence* $U_j \prod_{i=1}^{j-1}(1-U_i)$. So the convergence in distribution (7) of $|\mathcal{C}_n|/\sqrt{n}$ to L_1^0, and the "uniform random permutation" feature of the cyclic decomposition (Section 2.2(d)), combine to show that with *either* ordering. $|\mathcal{C}_{n,j}|/\sqrt{n} \stackrel{d}{\to} L^0_{I_j}$ with the same joint distribution:

$$(L^0_{I_j}, j \geq 1) \stackrel{d}{=} (L_1^0 U_j \prod_{i=1}^{j-1}(1-U_i),\ j \geq 1) \tag{9}$$

for both $I_j = I_j^D$ and $I_j = I_j^T$. This is part (iv) of Corollary 3, which is generalized later by (27) and Theorem 25.
(e) Let $\mathcal{B}_{n,(j)}$ be the jth largest basin of M_n, with some arbitrary convention for breaking ties, and let $\mathcal{C}_{n,(j)}$ be the cycle contained in $\mathcal{B}_{n,(j)}$. It follows immediately from the convergence in distribution (8) that

$$\left(\frac{|\mathcal{B}_{n,(j)}|}{n}, \frac{|\mathcal{C}_{n,(j)}|}{\sqrt{n}}\right) \stackrel{d}{\to} (\lambda_{I_{(j)}}, L^0_{I_{(j)}}) \tag{10}$$

jointly as j varies, where $I_{(j)}$ is the length-ranked interval partition derived from either either (I_j^D) or (I_j^T). This is part (i) of Corollary 3. By exchangeability considerations, before passage to the limit the bivariate sequence in

(8) as j varies is that in (10) biased by cycle-size in the cycles-first order and biased by basin-size in the basins-first ordering. Hence the conclusions of parts (ii) and (iii) of Corollary 3, by a straightforward passage to the limit.
(f) Due to Section 2.2(a), it makes no difference to anything if in the cycles-first ordering we replace the ordering within the jth cycle $c_{n,j,1}, c_{n,j,2}, \ldots, c_{n,j}$ by the possibly more natural $c_{n,j}, c_{n,j,1}, c_{n,j,2}, \ldots, c_{n,j,|\mathcal{C}_{n,j}|-1}$. But in the basins-first ordering, this innocent looking change would spoil convergence to $2|B^{\mathrm{br}}|$. This is because in the basins-first ordering the tree with root $c_{n,1}$ is the tree containing 1, which is a size-biased choice from the exchangeable random partition of $[n]$ into tree components. As such, it tends to be a big tree. In fact, results from [2] imply that, if the mapping-walk is started by the excursion coding the tree rooted at $c_{n,1}$, the limit process will start with a zero free interval whose length is distributed as $D_U - G_U$ in Lemma 9 below for $\alpha = \frac{1}{2}$. Such a process is obviously not $2|B^{\mathrm{br}}|$ or any other familiar Brownian process.
(g) The proof of Theorem 6 yields more information about the asymptotic sizes of tree components than can be deduced from the statement of that theorem. For instance, if $|\mathcal{T}_{n,(i)}|$ are the ranked sizes of the tree components of M_n, and $H_{n,i}$ are the corresponding maximal tree heights, as in (5), then $(|\mathcal{T}_{n,(i)}|/n, H_{n,i}/\sqrt{n})_{i=1,2,\ldots}$ converges in distribution to the sequence of ranked lengths and corresponding maximal heights of excursions of $2|B^{\mathrm{br}}|$, whose distribution was described in [35, Theorem 1 and Example 8]. If only the tree components of $\mathcal{B}_{n,j}$ were considered, the limit would be derived from excursions of B^{br} over the appropriate random interval I_j as in Theorem 6, with joint convergence as j varies.

3 The bridge decompositions

This section presents our main results for the D- and T-partitions. For ease of comparison, the results are presented together here, with outlines of the proofs. Some proofs and further details are deferred to Section 4 for the D-partition, and to Section 5 for the T-partition. Our primary interest is the analysis of the D- and T-partitions derived from a standard Brownian bridge, and the connections between these random partitions and the asymptotics of random mappings discussed in Section 2. But we find that our analysis applies just as well to the D- and T-partitions for a standardized bridge B^{br} derived from B a recurrent self-similar Markov process whose inverse local time process at 0 is a stable subordinator of index α for some $\alpha \in (0,1)$. Readers who don't care about this generalization can assume throughout this section that B is standard one-dimensional Brownian motion, and $\alpha = \beta = \frac{1}{2}$.

3.1 General framework

Following Pitman-Yor [35, §2], we make the following basic assumptions:

- $B := (B_t, t \geq 0)$ is a real or vector-valued strong Markov process, started at $B_0 = 0$, with state space a cone contained in $\mathbb{R}^d$ for some $d = 1, 2, \ldots$, and càdlàg paths.
- B is β-self-similar for some real β. That is to say, if $B_*[0,t]$ now denotes the standardized process derived from B on $[0,t]$ as in (3), using λ_I^β instead of $\sqrt{\lambda_I}$ in the denominator, then $B_*[0,t] \stackrel{d}{=} B[0,1]$ for all $t > 0$.
- The point 0 is a regular recurrent point for B, meaning that almost surely both 0 and ∞ are points of accumulation of the zero set of B.

As a well known consequence of these assumptions [14, 35], there exists a continuous local time process for B at 0, say $(L_t^0(B), t \geq 0)$, whose inverse process

$$\tau_\ell := \inf\{t : L_t^0(B) > \ell\} \qquad (\ell \geq 0)$$

is a stable subordinator of index α for some $\alpha \in (0,1)$. That is

$$E \exp(-\xi \tau_\ell) = \exp(-\ell c \xi^\alpha) \qquad (\xi \geq 0) \tag{11}$$

for some $c > 0$, in which case

$$L_t^0(B) = \frac{\Gamma(1-\alpha)}{c} \lim_{\varepsilon \to 0} \varepsilon^\alpha N_{t,\varepsilon}(B) \tag{12}$$

uniformly for bounded t almost surely, where $N_{t,\varepsilon}(B)$ is the number of excursion intervals of B in $[0,t]$ whose length is greater than ε. Formula (12) can be then used with X instead of B to define $L_t^0(X)$ for various other processes X derived from B by conditioning or scaling, such as the standardized bridge B^{br} introduced in the next paragraph. As a consequence of (12) with X instead of B, there is following basic *α-scaling rule* for such local time processes: for $I = [G_I, D_I]$ a random subinterval of length $\lambda_I := D_I - G_I$ contained in the time domain of X, and $L^0_I(X) := L^0_{D_I}(X) - L^0_{G_I}(X)$,

$$L^0_I(X) = \lambda_I^\alpha \, L^0_1(X_*[I]). \tag{13}$$

Associated with the self-similar Markov process B are corresponding distributions of a *standard B-bridge* B^{br}, a *standard B-excursion* B^{ex}, and a *standard B-meander* B^{me}, defined by the following identities in distribution, valid for all $t > 0$:

$$B_*[0, G_t] \stackrel{d}{=} B^{\mathrm{br}}; \quad B_*[G_t, D_t] \stackrel{d}{=} B^{\mathrm{ex}}; \quad B_*[G_t, t] \stackrel{d}{=} B^{\mathrm{me}} \tag{14}$$

where $G_t := G_t(B)$, $D_t := D_t(B)$, and for any process X we use the notation

$$G_t(X) := \sup\{u < t : X_u = 0\} \tag{15}$$
$$D_t(X) := \inf\{u \geq t : X_u = 0\}. \tag{16}$$

See [9] for a review of properties of B^{br}, B^{ex} and B^{me} in the *Brownian case* when B is Brownian motion with state space $\mathbb{R}$, and $\beta = \alpha = \frac{1}{2}$. See [25, §3] and [34] for some treatment of B^{br} and B^{ex} in the *Bessel case* when B with state space $\mathbb{R}_{\geq 0}$ is a recurrent Bessel process of dimension $\delta = 2 - 2\alpha \in (0, 2)$, and $\beta = \frac{1}{2}$. Other examples are provided by recurrent stable Lévy processes [8], symmetrized or skew Bessel processes [42], and Walsh processes [6, 7].

According to the Lévy-Itô theory of excursions of B, applied to the standard B-bridge as in [31, 35], if (I_j^{ex}) is the interval partition of $[0, 1]$ defined by the excursion intervals of B^{br} in length-ranked order, then the processes $B_*^{\mathrm{br}}[I_j^{\mathrm{ex}}]$ are i.i.d. copies of B^{ex}, independent of (I_j^{ex}), which is an *exchangeable interval partition* in the sense of [16] recalled in Section 6.2. Moreover, the distribution of ranked lengths $(\lambda_{I_j^{\mathrm{ex}}})$ depends only on α, as described in [32, (16)] and [35, Example 8]. This general excursion decomposition of B^{br} implies that various results known for Bessel bridges hold also in the present general setting, and we take this for granted without further comment.

3.2 Main Results

All results of this section are presented with the notation and general framework of the previous section: B^{br} is the standard B-bridge derived from a self-similar recurrent Markov process B whose continuous local time process $(L^0_t(B), t \geq 0)$ is the inverse of a stable subordinator $(\tau_\ell, \ell \geq 0)$ of index

$\alpha \in (0,1)$. The D- and T-partitions are defined in terms of B^{br} and its local time process at 0, according to Definitions 1 and 2.

Corollary 3, presented in the introduction in the Brownian case, is true in the more general framework of this section, as a consequence of the following theorem:

Theorem 7 *For a random interval $I \subseteq [0,1]$, let μ_I denote the random occupation measure induced by the path of $B^{\mathrm{br}}[I]$, so each Borel subset A of the state space of B^{br}*

$$\mu_I(A) := \int_I 1(B_t^{\mathrm{br}} \in A)dt.$$

(i) *The sequence of occupation measures (μ_{I_j}) has the same distribution for each of the two length-ranked partitions $(I_j) = (I_{(j)}^D)$ and $(I_j) = (I_{(j)}^T)$.*
(ii) *For $(I_j) = (I_j^D)$ the sequence of occupation measures (μ_{I_j}) is in λ-biased order, where λ_{I_j} is the total mass of the random measure μ_{I_j}.*
(iii) *For $(I_j) = (I_j^T)$ the sequence of occupation measures (μ_{I_j}) is in L^0-biased order, where $L_{I_j}^0 := L_{I_j}^0(B^{\mathrm{br}})$.*
(iv) *For (I_j) equal to any one of the four interval partitions (I_j^D), $(I_{(j)}^D)$, (I_j^T) or $(I_{(j)}^T)$, conditionally given $\lambda_{I_j} = \lambda_j$ and $L_{I_j}^0 = \ell_j$ for all $j = 1,2,\ldots$, the random occupation measures $\mu_{I_j}, j = 1,2,\ldots$ are independent, with μ_{I_j} distributed like the random occupation measure of a process with the common conditional distribution of*

$$(B[0,t] \,|\, B_t = 0, L_t^0 = \ell) \stackrel{d}{=} (B[0,\tau_\ell] \,|\, \tau_\ell = t) \tag{17}$$

for $t = \lambda_j$ and $\ell = \ell_j$.

Proof. Propositions 10, 11 and 14 provide more explicit descriptions of the law of $(\lambda_{I_j}, L_{I_j}^0, B_*^{\mathrm{br}}[I_j])_{j=1,2,\ldots}$, for each of the four interval partitions (I_j). The above results for occupation measures are deduced from these propositions using Lemma 13. The fundamental *switching identity* (17) is well known [31, §5]. □

By general theory of local time processes for diffusions or continuous semimartingales [15, 39, 38], in the Brownian and Bessel cases for each random subinterval I of $[0,1]$ the random occupation measure μ_I derived from B^{br}

has an almost surely continuous density L_I^x relative to m at x, where m is a multiple of the speed measure of the one-dimensional diffusion B. To be precise about normalization of local times, in the Brownian case with state space $\mathbb{R}$, we take $m(dx) = dx$, so that (11) holds with $\alpha = \frac{1}{2}$ and $c = \sqrt{2}$. In the Bessel(δ) case with state space $\mathbb{R}_{\geq 0}$, we take $m(dx) = 2x^{\delta-1}dx$, so that (11) holds with $\alpha = 1 - \delta/2$ and $c = 2^{1-\alpha}\Gamma(1-\alpha)/\Gamma(\alpha)$, by [31, (7.c)]. In either case, $L^0_{I_j}$ in (iii) and (iv) is recovered like λ_{I_j} as a measurable function of the random occupation measure μ_{I_j}. The distribution of the local time density of the conditional occupation measure in (iv) is described by a conditional form of the Ray-Knight theorem: see [20, 28] for details in the Brownian case.

Our analysis of the D-partition is the following expression of the decomposition of B^{br} at the times D_{V_j}, implicit in [2] in the Brownian case:

Lemma 8 [2] *For each j, the pre-D_{V_j} fragment of the bridge $B^{\mathrm{br}}[0, D_{V_j}]$ is independent of the standardized post-D_{V_j} fragment $B^{\mathrm{br}}_*[D_{V_j}, 1]$, which has the same distribution as B^{br}.*

This is easily verified, because the D_{V_j} are stopping times relative to a filtration with respect to which B^{br} has a strong Markov property.

To describe various distributions, let $(\Gamma_s, s \geq 0)$ denote a *standard gamma process*, that is the increasing Lévy process with marginal densities

$$P(\Gamma_s \in dx)/dx = \frac{1}{\Gamma(s)}\, x^{s-1}\, e^{-x} \qquad (x > 0), \tag{18}$$

so $\Gamma_t - \Gamma_s \stackrel{d}{=} \Gamma_{t-s}$ for $0 < s < t$. Recall that for $a, b > 0$ the beta(a, b) distribution is that of

$$\beta_{a,b} := \Gamma_a/\Gamma_{a+b}, \text{ which is independent of } \Gamma_{a+b}, \text{ with} \tag{19}$$

$$P(\beta_{a,b} \in du) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}du \qquad (0 < u < 1). \tag{20}$$

It is well known [25, Lemma 3.7] that for $G_t = G_t(B)$,

$$\text{the standard } B\text{-bridge } B_*[0, G_t] \text{ is independent of } G_t \text{ with } G_t/t \stackrel{d}{=} \beta_{\alpha,1-\alpha}. \tag{21}$$

Lemma 9 [2, Prop. 2], [27, Prop. 15] *Let U with uniform$[0,1]$ distribution be independent of B^{br}, and let $G_U := G_U(B^{\mathrm{br}})$, $D_U := D_U(B^{\mathrm{br}})$. Then*

$$(G_U, D_U - G_U, 1 - D_U) \stackrel{d}{=} (,\ {}_{\alpha},,\ {}_{1} - ,\ {}_{\alpha},,\ {}_{1+\alpha} - ,\ {}_{1})/,\ {}_{1+\alpha}.$$

Moreover, the random vector $(G_U, D_U - G_U, 1 - D_U)$ and the three standardized processes $B_*^{\mathrm{br}}[0, G_U]$, $B_*^{\mathrm{br}}[G_U, D_U]$ and $B_*^{\mathrm{br}}[D_U, 1]$ are independent, with

$$B_*^{\mathrm{br}}[0, G_U] \stackrel{d}{=} B_*^{\mathrm{br}}[D_U, 1] \stackrel{d}{=} B^{\mathrm{br}} \quad \text{and} \quad B_*^{\mathrm{br}}[G_U, D_U] \stackrel{d}{=} B^{\mathrm{ex}}. \tag{22}$$

Proposition 10 *For the D-partition*
(i) *the sequence of lengths is such that*

$$\lambda_{I_j^D} = W_j \prod_{i=1}^{j-1} (1 - W_i) \tag{23}$$

for a sequence of independent random variables W_j with $W_j \stackrel{d}{=} \beta_{1,\alpha}$.
(ii) *The corresponding sequence of local times at 0 can be expressed as*

$$L^0_{I_j^D} = \lambda^{\alpha}_{I_j^D} L^0_1(B_*^{\mathrm{br}}[I_j^D]) \tag{24}$$

where the $L^0_1(B_*^{\mathrm{br}}[I_j^D])$ are independent random variables, independent also of the lengths $\lambda_{I_j^D}$, with

$$L^0_1(B_*^{\mathrm{br}}[I_j^D]) \stackrel{d}{=} L^0_1(B) \stackrel{d}{=} \tau_1^{-\alpha} \tag{25}$$

for τ_1 with the stable distribution of index α defined by (11).
(iii) *The standardized path fragments $B_*^{\mathrm{br}}[I_j^D]$ are independent and identically distributed like $B_*^{\mathrm{br}}[0, D_U]$, and independent of the sequence of lengths $(\lambda_{I_j^D})$.*
(iv) *For the length-ranked D-intervals $I_{(j)}^D$ instead of I_j^D, the lengths $(\lambda_{I_{(j)}^D})$ have the Poisson-Dirichlet(α) distribution defined by ranking $(\lambda_{I_j^D})$ as in* (i), *while parts* (ii) *and* (iii) *hold without change.*

Proof. Parts (i)-(iii) are obtained by repeated application of Lemmas 8 and 9, using the α-scaling rule (13) for local times and (21), as in [25, Lemma 3.11], for part (ii). The second identity in (ii) is a well-known consequence of the inverse relation between $(L_t^0(B), t \geq 0)$ and $(\tau_\ell, \ell \geq 0)$, as discussed in [31]. Part (iv) follows immediately from (i)-(iii). $\square$

See [18, 32] and Lemma 15 for background on the Poisson-Dirichlet distribution appearing in (iv). Lévy [21] showed that in the Brownian case the common distribution of $L_1^0(B)$ and $\tau_1^{-1/2}$ appearing in (25) is simply the distribution of $|B_1|$, with B_1 standard Gaussian. But this does not generalize to the Bessel(δ) case for general $\delta = 2 - 2\alpha$. Then $B_1 \stackrel{d}{=} \sqrt{2, {}_{1-\alpha}}$, which is a simple transformation of the stable(α) distribution of τ_1 only for $\alpha = \frac{1}{2}$.

The difficulty involved in Theorem 7 is that Definition 2 of the T_j involves the local time $L_1^0 := L_1^0(B^{\mathrm{br}})$, which depends on the path of B^{br} over the whole interval $[0,1]$. While we can describe the finite-dimensional distributions of the bivariate sequence $(\lambda_{I_j^T}, L^0_{I_j^T})_{j=1,2,\ldots}$ by conditioning on L_1^0 (see Proposition 20), this description is more complicated than our description of $(\lambda_{I_j^D}, L^0_{I_j^D})_{j=1,2,\ldots}$ in Proposition 10. In particular,

$$\lambda_{I_1^T} \stackrel{d}{\neq} \lambda_{I_1^D}. \tag{26}$$

Indeed, by (23) we have

$$E(\lambda_{I_1^D}) = E(W_1) = 1/(1+\alpha) > 1/2,$$

whereas (by symmetry of B^{br} with respect to time reversal in the Brownian or Bessel case) the distribution of T_1 is symmetric about $\frac{1}{2}$, so whatever $\alpha \in (0,1)$

$$E(\lambda_{I_1^T}) = E(T_1) = 1/2.$$

Still, as explained combinatorially in the Brownian case around (9), the two partitions give rise to the same distribution for the sequence of local times:

$$\left(L^0_{I_j^D}\right) \stackrel{d}{=} \left(L^0_{I_j^T}\right) := \left(L_1^0\, U_j \prod_{i=1}^{j-1}(1-U_i)\right) \tag{27}$$

where the second equality by definition is read from (1). The first equality in distribution of sequences follows from Lemma 8 and the consequence of Lemma 9, noted in [2, (3)-(4)] in the Brownian case, that

$$L^0_{I_1^D}/L_1^0 \text{ has uniform distribution on } (0,1), \text{ and is independent of } L_1^0. \tag{28}$$

As indicated in Section 6.2, this can also be checked in general using the exchangeability of the excursion interval partition.

According to Proposition 10, the standardized bridge fragments over intervals of the D-partition are i.i.d. copies of $B_*^{\mathrm{br}}[0, D_U]$, both for the intervals in their original order and for the intervals in length-ranked order. A subtle feature of the T-partition is that the standardized bridge fragments over its intervals are neither independent nor identically distributed in their original order, but these fragments become i.i.d. when put into length-ranked order. This and other parallels between the T- and D-partitions in length-ranked order are presented in the following Proposition:

Proposition 11 *For the T-partition in length-ranked order*
(i) *the sequence of lengths $(\lambda_{I^T_{(j)}})$ has the same Poisson-Dirichlet(α) distribution as $(\lambda_{I^D_{(j)}})$.*
(ii) *The corresponding sequence of local times at 0 can be expressed as*

$$L^0_{I^T_{(j)}} = \lambda^\alpha_{I^T_{(j)}} \, L^0_1(B^{\mathrm{br}}_*[I^T_{(j)}]) \tag{29}$$

where the $L^0_1(B^{\mathrm{br}}_*[I^T_{(j)}])$ are independent random variables, independent also of the lengths $L^0_{I^T_{(j)}}$, with

$$L^0_1(B^{\mathrm{br}}_*[I^T_{(j)}]) \stackrel{d}{=} L^0_1(B) \stackrel{d}{=} \tau_1^{-\alpha} \tag{30}$$

just as in (25).
(iii) *The standardized path fragments $B^{\mathrm{br}}_*[I^T_{(j)}]$ are independent and identically distributed like $B_*[0,\tau_1]$, and independent of the sequence of lengths $(\lambda_{I^T_{(j)}})$.*

The only difference between this description of the law of the sequence $(\lambda_{I_j}, L^0_{I_j}, B^{\mathrm{br}}_*[I_j])_{j=1,2,\ldots}$ for $I_j = I^T_{(j)}$, and the previous description in Proposition 10 for $I_j = I^D_{(j)}$, is that the common distribution of the standardized T-fragments is that of $B^{\mathrm{br}}_*[0, D_U]$, whereas the common distribution of the standardized D-fragments is that of $B_*[0,\tau_1]$. The standardized process $B_*[0,\tau_1]$ is known as the *pseudo-bridge* associated with the self-similar Markov process B. The following Lemma was established by Biane, Le Gall and Yor [11] in the Brownian case, and extended to the Bessel case in [31, Theorem 5.3].

Lemma 12 [11],[31]. *The law of the pseudo-bridge $B_*[0,\tau_1]$ is mutually absolutely continuous with respect to the law of $B^{\rm br}$, with density proportional to $1/L_1^0(B)$ relative to the law of $B^{\rm br}$. That is, for all non-negative measurable path functionals F*

$$E[F([B_*[0,\tau_1])] = \frac{1}{c\alpha,\,(\alpha)} E\left[\frac{F(B^{\rm br})}{L_1^0(B^{\rm br})}\right].$$

where c is determined by the normalization of local time via (11)*.*

While the laws of $B_*^{\rm br}[0,D_U]$ and the pseudo-bridge $B_*[0,\tau_1]$ are mutually singular, their random occupation measures have the same distributions. In fact, the sample path of $B_*^{\rm br}[0,D_U]$ is simply a random rearrangement of the sample path of $B_*[0,\tau_1]$:

Lemma 13 *Let U be a uniform $(0,1)$ variable independent of $B^{\rm br}$, and independent of X distributed like $B_*[0,\tau_1]$. Then a process Y distributed like $B_*^{\rm br}[0,D_U]$ is created by the following rearrangement of the path of X, whereby the random occupation measures of X and Y are pathwise identical: let (G_U,D_U) be the excursion interval of X straddling time U, and let Y be derived from X by swapping the order of the path fragments $X[G_U,D_U]$ and $X[D_U,1]$, say*

$$Y = X[0,G_U] : X[D_U,1] : X[G_U,D_U] \tag{31}$$

with an obvious notation for concatenation of path fragments.

Proof. By construction, the path of Y ends with a B-excursion of length $1-G_1(Y) = D_U - G_U$. The joint law of $Y[0,G_1(Y)]$ and $Y[G_1(Y),1] := X[G_U,D_U]$ was described in [31, Theorem 1.3] and [25, Theorem 3.1 and (3.d)], and is identical to the joint law of $Z[0,G_1(Z)]$ and $Z[G_1(Z),1]$ for $Z := B_*^{\rm br}[0,D_U(B^{\rm br})]$, which can be read from Lemma 9. To be explicit, the common distribution of $Y[0,G_1(Y)]$ and $Z[0,G_1(Z)]$ is that of $B[0,G_1(B)]$ described by (21), while both $Y_*[G_1(Y),1] := X_*[G_U,D_U]$ and $Z_*[G_1(Y),1] := B_*^{\rm br}[G_U(B^{\rm br}),D_U(B^{\rm br})]$ are standard B-excursions. Since the excursion is in each case independent of the preceding fragment, it follows that $Y \stackrel{d}{=} Z$. □

Proposition 14 *Fix $\xi > 0$. Let G be a random variable independent of $B^{\rm br}$, with $G \stackrel{d}{=} \,,_\alpha/\xi$. The distributions of the two bivariate sequences, defined by*

the lengths and bridge local time measures of intervals of the D-partition and the T-partition respectively, are determined as follows:
(i) *For $I_j = I_j^D$ the bivariate sequence*

$$(G\lambda_{I_j}, G^\alpha L_{I_j}^{\mathrm{br}})_{j=1,2,\ldots} \tag{32}$$

is the sequence of points (X_j, Y_j), in X-biased random order, of a Poisson process on $\mathbb{R}^2_{>0}$ with intensity measure

$$\nu(dt, d\ell) := \alpha t^{-1} e^{-\xi t} dt P(t^\alpha \tau_1^{-\alpha} \in d\ell) = \ell^{-1} P(\tau_\ell \in dt) e^{-\xi t} \tag{33}$$

for τ_1 as in (11), which makes

$$\Sigma_j X_j \stackrel{d}{=} \frac{\Gamma_\alpha}{\xi} \text{ and } \Sigma_j Y_j \stackrel{d}{=} \frac{\Gamma_1}{c\xi^\alpha}. \tag{34}$$

(ii) *If the points (X_j, Y_j) of a Poisson process with intensity ν on $\mathbb{R}^2_{>0}$ are listed in X-biased order then*

$$(\lambda_{I_j^D}, L^0_{I_j^D})_{j=1,2,\ldots} \stackrel{d}{=} \left(\frac{X_j}{\Sigma_X}, \frac{Y_j}{\Sigma_X^\alpha}\right)_{j=1,2,\ldots} \tag{35}$$

for $\Sigma_X := \sum_j X_j$ as in (34).
(iii) *For $I_j = I_j^T$ the bivariate sequence in* (32) *is the sequence of points, say (X_j', Y_j'), in Y'-biased random order, of another Poisson process on $\mathbb{R}^2_{>0}$ with the same intensity measure ν. So if in* (ii) *the points (X_j, Y_j) are listed instead in Y-biased order, then* (35) *holds with the sequence of T-intervals instead of the sequence of D-intervals.*

Proof. Part (i) is proved in Section 4. Part (ii) is just a restatement of part (i). Part (iii) is proved in Section 5. □

Note that the normalization in (35) involves Σ_X and its αth power, both for the D-partition and for the T-partition. Obviously, this is easier to handle if the sampling is X-biased rather than Y-biased, which is one explanation of why various distributions associated with (I_j^D) are simpler than their counterparts for (I_j^T).

4 Analysis of the D-partition

As a preliminary for the proof of Proposition 14 (i), we recall the following well known lemma, which characterizes the distribution of a sequence (Q_j), known as the $GEM(\theta)$ distribution after Griffiths, Engen and McCloskey. The distribution of $(Q_{(j)})$ obtained by ranking (Q_j) is known as the *Poisson-Dirichlet distribution with parameter* θ. See [17], [18, §9.6], [32].

Lemma 15 (Characterizations of $GEM(\theta)$ [23], [25], [26]) *Fix* $\theta > 0$ *and* $\xi > 0$. *Let* G *and* $Q_j, j = 1, 2, \ldots$ *be non-negative random variables. Then the following are equivalent:*
(i) *the sequence* (Q_j) *admits the representation* $Q_j = W_j \prod_{i=1}^{j-1}(1 - W_i)$ *where the* W_j *are independent beta*$(1,\theta)$ *variables, and* G *is independent of* (Q_j) *with* $G \stackrel{d}{=} ,_{\theta}/\xi$;
(ii) $\sum_j Q_j = 1$ *a.s. and* (GQ_j) *is the sequence of points of a Poisson point process on* $\mathbb{R}_{>0}$ *with intensity* $\theta t^{-1} e^{-\xi t} dt$, *listed in size-biased order.*

The next well known result [18, §5.2] , [37, Prop. 4.10.1], combined with the previous lemma, provides an efficient way to identify various Poisson processes.

Lemma 16 (Poisson marking)) *Let* $(S, \mathcal{S})$ *and* $(T, \mathcal{T})$ *be two measurable spaces. Let* (X_j) *and* (Y_j) *be two sequences of random variables, with values in* S *and* T *respectively, such that the counting process* $\sum_j 1(X_j \in \cdot)$ *is Poisson with intensity measure* μ *on* $\mathcal{S}$, *and the* Y_j *are conditionally independent given* (X_j), *with*

$$P(Y_j \in \cdot \,|\, X_1, X_2, \ldots) = P'(X_j, \cdot)$$

for some Markov kernel P' *from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$. *Then the counting process* $\sum_j 1((X_j, Y_j) \in \cdot)$ *is a Poisson process on the product space* $S \times T$ *with intensity measure* $\mu(dx)P'(x, dy)$ *on the product* σ*-field.*

Proof of Proposition 14 (i) Proposition 10(i) and Lemma 15 (i) show that $(\lambda_{I_j^D}, j \geq 1)$ has $GEM(\alpha)$ distribution. By assumption, G is independent of this sequence with $G \stackrel{d}{=} ,_{\alpha}/\xi$. Lemma 15 implies that $(G\lambda_{I_j^D})$ is the size-biased ordering of a Poisson point process of intensity $\alpha t^{-1} e^{-\xi t} dt$. Proposition 10(ii) and Lemma 16 now identify the $(G\lambda_{I_j^D}, G^\alpha L^{\mathrm{br}}_{I_j^D})$ as the points of a Poisson process with intensity measure ν defined by the first expression in (33). To check the equality of the two expressions in (33), let

$$f_\ell(t) := P(\tau_\ell \in dt)/dt. \tag{36}$$

Since $\tau_\ell \stackrel{d}{=} \ell^{1/\alpha}\tau_1$ by (11),

$$f_\ell(t) = \ell^{-1/\alpha} f_1(t/\ell^{1/\alpha}) \tag{37}$$

whereas by another change of variables

$$P(t^\alpha \tau_1^{-\alpha} \in d\ell)/d\ell = \alpha^{-1} t \ell^{-1-1/\alpha} f_1(t/\ell^{1/\alpha}) = \alpha^{-1} t \ell^{-1} f_\ell(t) \tag{38}$$

and the identity follows. By application of (11), the ℓ-marginal of ν is $\ell^{-1} e^{-c\xi^\alpha \ell} d\ell$. The distribution of $\sum_j Y_j$ is the infinitely divisible law with this Lévy measure, that is the exponential distribution with rate $c\xi^\alpha$. □

Implicit in Lemma 12 and (38) is the following formula of [25, (3.u)] for the density of $L_1^0 := L_1^0(B^{\mathrm{br}})$

$$P(L_1^0 \in d\ell) = c\alpha, (\alpha)\ell P(\tau_1^{-\alpha} \in d\ell) = c, (\alpha) f_\ell(1) d\ell \tag{39}$$

for $f_\ell(x)$ as in (36) the stable(α) density of τ_ℓ determined by (11). That is to say, the distribution of $L_1^0(B^{\mathrm{br}})$ is obtained by size-biasing the common distribution of $\tau_1^{-\alpha}$ and $L_1^0(B)$. In particular, the general formula (39) is consistent with Lévy's well known formulae in the Brownian case [21], with $\alpha = \frac{1}{2}, c = \sqrt{2}$

$$f_\ell(x) = \frac{\ell}{\sqrt{2\pi}} x^{-3/2} e^{-\frac{1}{2}\ell^2/x} \tag{40}$$

and

$$P(L_1^0 \in d\ell)/d\ell = \ell e^{-\frac{1}{2}\ell^2}. \tag{41}$$

For general α, a series expression for $f_\ell(x)$ is known [36, 43, 44, 40]. If $\alpha = 1/n$ for some $n = 2, 3, \ldots$, integral expressions for $f_\ell(x)$ can be derived from a representation of $1/\tau_\ell$ as a product of $n-1$ independent gamma variables [43, Theorem 3.4.3]. To conclude this section, we record the following immediate consequence of Proposition 14(i):

Corollary 17 *Let (σ_j) be a sequence of i.i.d. copies of τ_1 with the stable (α) distribution* (11)*, and let (Q_j) with $GEM(\alpha)$ distribution of Lemma 15 be independent of (σ_j). Let $L_j := (Q_j/\sigma_j)^\alpha$ and $L := \sum_j L_j$. Then $L \stackrel{d}{=} L_1^0$ as in* (39)*, and the sequence (L_j/L) has $GEM(1)$ distribution, independently of L.*

5 Analysis of the T-partition

We start by recalling the structure of a Markov process up to the last time it visits its initial state before an independent exponential time. This does not involve the self-similarity assumption.

Lemma 18 [14] *Let* $(\tau_\ell, \ell \geq 0)$ *be a drift-free subordinator which is the inverse of the continuous local time process* $(L_t^0(B), t \geq 0)$ *of a regular recurrent point* 0*, for a strong Markov process* B *started at* 0*. Let* ε *be an exponential variable with rate* ξ*, with* ε *independent of* B*, and let*

$$G := G_\varepsilon(B) \text{ and } L := L_G^0(B) = L_\varepsilon^0(B). \tag{42}$$

(i) *The local time* L *has exponential distribution with rate* $\psi(\xi)$*, the Laplace exponent of the subordinator defined by* $E(e^{-\xi\tau_\ell}) = e^{-\psi(\xi)\ell}$.
(ii) *For* $\ell > 0$*, there is the equality in distribution of path fragments*

$$(B[0,G] \,|\, L = \ell) \stackrel{d}{=} (B[0,\tau_\ell] \,|\, \tau_\ell < \varepsilon). \tag{43}$$

(iii) *The joint distribution of* (G, L) *is*

$$P(G \in dt, L \in d\ell) = \psi(\xi) d\ell e^{-\xi t} P(\tau_\ell \in dt). \tag{44}$$

which is the distribution of the value at time 1 *of a drift free bivariate subordinator with Lévy measure*

$$\nu(dt, d\ell) = \ell^{-1} d\ell e^{-\xi t} P(\tau_\ell \in dt) \tag{45}$$

whose ℓ*-marginal is the Lévy measure* $\ell^{-1} e^{-\psi(\xi)\ell} d\ell$ *of the exponential distribution of* L*.*

Proof. These results are derived from Itô's theory of excursions of B, by letting $(N_t, t \geq 0)$ be a Poisson process with rate ξ, independent of B, and taking ε to be the time of the first point of N. To briefly recall the argument, say that a jump interval (τ_{y-}, τ_y) of the inverse local time process τ is *marked* if $N(\tau_{y-}, \tau_y] > 0$ and *unmarked* otherwise. Then, by basic theory of Poisson point processes, the sum of unmarked jumps

$$\tau_\ell^{\mathrm{u}} := \sum_{0<y<\ell} (\tau_y - \tau_{y-}) 1(N(\tau_{y-}, \tau_y] = 0) \tag{46}$$

defines a subordinator with distribution

$$P(\tau_\ell^{\mathrm{u}} \in dt) = e^{\psi(\xi)\ell - \xi t} P(\tau_\ell \in dt). \tag{47}$$

The left end G of the first marked interval is $G = \tau_{L-} = \tau_L^{\mathrm{u}}$, and the subordinator τ^{u} summing unmarked jumps of τ is independent of L, the local time of the first marked jump. See also [14, 31, 35, 39]. □

To be more explicit, part (iii) of the Lemma states that

$$(G, L) \stackrel{d}{=} (\Sigma_j X_j, \Sigma_j Y_j) \tag{48}$$

for (X_j, Y_j) the points of a Poisson point process on $\mathbb{R}^2_{>0}$ with intensity measure ν defined by (45). In particular, for a self-similar B as in Section 3, this measure ν is identical to the measure ν featured in (33).

In the setting of Lemma 18, even with construction of the Poisson process of marks of rate ξ independent of B, more randomization is required to construct points (X_j, Y_j) such that (48) holds with equality almost surely rather than just in distribution. But this can be done by the following construction, which is the basis of our proofs of Proposition 11 and Proposition 14 (iii).

Lemma 19 *In the setting of Lemma 18, let $I := [G_I, D_I]$ be a random subinterval of $[0, 1)$, where the endpoint 1 is deliberately excluded, to avoid the jump of the inverse local time process (τ_ℓ) at time $\ell = L$ in the following construction. Suppose I is independent of B and ε, and define further random intervals*

$$IL := [G_I L, D_I L] \quad \textit{and} \quad \tau(IL) := [\tau(G_I L), \tau(D_I L)] \tag{49}$$

where $\tau(\ell) := \tau_\ell$ for $\ell \geq 0$.
(i) *For $y > 0$, there is the equality in distribution of path fragments*

$$(B[\tau(IL)] \,|\, \lambda_{IL} = y) \stackrel{d}{=} (B[0, \tau_y] \,|\, \tau_y < \varepsilon) \tag{50}$$

where $\lambda_{IL} := D_I L - G_I L$ is the increment of local time of B over the time interval $\tau(IL)$.
(ii) *If (I_j) is an interval partition of $[0, 1)$ which is independent of ε and B, and $(\tau_{I_j L})$ is the corresponding interval partition of $[0, G)$, then given the sequence of local time increments $(\lambda_{I_j L})$ the path fragments $B[\tau_{I_j L}]$ are*

conditionally independent with distributions described by (50) *for* I_j *instead of* I.
(iii) *If* $I_j := [\hat{V}_{j-1}, \hat{V}_j]$ *with* $\hat{V}_j := 1-\prod_{i=1}^{j}(1-U_i)$ *for independent uniform*$(0,1)$ *variables* U_i *independent of* B *and* ε, *then the bivariate sequence of local time increments and path fragments*

$$(\lambda_{I_j L}, B[\tau(I_j, L)])_{j=1,2,\ldots} \tag{51}$$

is the sequence of points of a Poisson point process on $\mathbb{R}_{>0} \times \Omega$, *in local-time biased order, for a suitable space of path fragments* Ω *of arbitrary finite length, with intensity measure*

$$\ell^{-1} P(\tau_\ell < \varepsilon, B[0, \tau_\ell] \in d\omega) \tag{52}$$

whose ℓ*-marginal is the Lévy measure* $\ell^{-1}e^{-\psi(\xi)\ell}d\ell$ *of the exponential distribution of* L *with rate* $\psi(\xi)$.
(iv) *Let* $X_j := \tau(G_{I_j}L) - \tau(G_{I_j}L)$ *be the length and* $Y_j := \lambda_{I_j L}$ *the local time increment associated with the random subinterval* $\tau(I_j L)$ *of* $[0, G)$. *Then the* (X_j, Y_j) *are the points of a Poisson point process on* $\mathbb{R}^2_{>0}$ *with intensity measure* ν *defined by* (45), *in* Y*-biased random order, and*

$$(G, L) = (\Sigma_j X_j, \Sigma_j Y_j) \text{ almost surely.} \tag{53}$$

Proof. The first two assertions are straightforward consequences of the previous Lemma. Part (iii) follows from (ii), the Poisson representation of $GEM(1)$ in Lemma 15, and Poisson marking. Part (iv) follows from (iii) and Lemma 18(iii). □

Proof of Proposition 14 (iii) We will exploit the following construction of the standard bridge $B^{\rm br}$ of the self-similar Markov process B by random scaling, as in [31] and [35, Lemma 4]. Let

$$B^{\rm br} := B_*[0, G_\varepsilon(B)] \text{ for } \varepsilon \text{ independent of } B \text{ with } \varepsilon \stackrel{d}{=} \Gamma_1 \xi, \tag{54}$$

so ε is exponential with rate ξ. Then

$$G := G_\varepsilon(B) \stackrel{d}{=} \frac{\Gamma_\alpha}{\xi} \text{ and } L := L^0_G(B) = G^\alpha L^0_1(B^{\rm br}) \stackrel{d}{=} \frac{\Gamma_1}{c\xi^\alpha} \tag{55}$$

by (21), (19), and α-scaling of local times, where the exponential distribution of L is read from Lemma 18. Suppose now that $I_j := [\hat{V}_{j-1}, \hat{V}_j]$, for $\hat{V}_j$ as in

Lemma 19. The T-sequence is now constructed as a function of these $\hat{V}_j$ and $B^{\rm br} := B_*[0, G]$ as in (54) according to Definition 2, that is

$$T_j := \inf\{u : L_u^0/LB_1 > \hat{V}_j\}. \tag{56}$$

By 56 and (55),

$$\begin{aligned} \lambda_{I_j L} &= (\hat{V}_j - \hat{V}_{j-1})L = (\hat{V}_j - \hat{V}_{j-1})\, G^\alpha L_1^0(B^{\rm br}) = G^\alpha L^0_{I_j^T}(B^{\rm br}) \\ \lambda_{\tau_{I_j L}} &= \tau_{\hat{V}_j L} - \tau_{\hat{V}_{j-1} L} = G\, \lambda_{I_j^T} \\ B_*[\tau_{I_j L}] &= B_*^{\rm br}[I_j^T]. \end{aligned}$$

Part (iii) of Proposition 14 can now be read from Lemma 19 (iv). □

Proof of Proposition 11 Parts (i) and (ii) follow immediately from the result of Proposition 14(iii) proved above. Turning to consideration of the path fragments, we observe by switching identity (17) that

$$(B_*[0, \tau_\ell] \,|\, \tau_\ell = t) \stackrel{d}{=} (B^{\rm br} \,|\, L_1^0 = \ell t^{-\alpha}) \tag{57}$$

where $L_1^0 := L_1^0(B^{\rm br})$ as usual, and a regular conditional distribution for $B^{\rm br}$ given L_1^0 can be as constructed in [28, Lemma 12]. Hence from (52), if Ω_1 denotes a suitable space of paths of length 1, the trivariate sequence of local time increments, lengths of path fragments, and standardized path fragments

$$(\lambda_{I_j L}, \lambda_{\tau_{I_j L}}, B_*[\tau_{I_j L}])_{j=1,2,\ldots} = (G^\alpha L^0_{I_j^T}(B^{\rm br}), G\lambda_{I_j^T}, B_*^{\rm br}[I_j^T])_{j=1,2,\ldots} \tag{58}$$

is a Poisson process on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \Omega_1$ whose intensity measure is

$$\ell^{-1} d\ell\, P(\tau_\ell \in dt) e^{-\xi t} P(B^{\rm br} \in d\omega_1 \,|\, L_1^0 = \ell t^{-\alpha}). \tag{59}$$

Using the first form of ν in (33) to integrate out ℓ in (59), we see that the lengths and standardized fragments

$$(\lambda_{\tau_{I_j L}}, B_*[\tau_{I_j L}])_{j=1,2,\ldots} = (G\lambda_{I_j^T}, B_*^{\rm br}[I_j^T])_{j=1,2,\ldots} \tag{60}$$

form a Poisson process on $\mathbb{R}_{>0} \times \Omega_1$ whose intensity measure is

$$\alpha t^{-1} e^{-\xi t} dt Q(d\omega_1) \tag{61}$$

where

$$Q(d\omega_1) = \int_0^\infty P(B^{\rm br} \in d\omega_1 \,|\, L_1^0 = y) P(\tau_1^{-\alpha} \in dy) = P(B_*[0, \tau_1] \in d\omega) \tag{62}$$

by the switching identity (57). The factorization in (61) shows that the $B_*^{\rm br}[I_j^T]$ are i.i.d. copies of $B_*[0, \tau_1]$ when listed in length-ranked order. That is part (iii) of Proposition 11. □

5.1 Further distributional results

We record in this section a number of further formulae related to the distribution of the lengths and local times defined by the T-partition.

Proposition 20 *For the T-partition the $(2n+1)$-variate joint density of the total bridge local time L_1^0, the lengths of the first n intervals, and the local times at 0 on these intervals, is given by the formula*

$$P(L_1^0 \in d\ell, \lambda_{I_j^T} \in dx_j, L^{\mathrm{br}}(I_j^T) \in dy_j, 1 \leq j \leq n) =$$

$$c,(\alpha) d\ell\, f_{\ell-y_1-\cdots-y_n}(1-x_1-\cdots-x_n) \prod_{j=1}^{n} \frac{dx_i\, dy_j f_{y_j}(x_j)}{\ell - y_1 - \cdots - y_{j-1}}$$

for $f_y(x) := P(\tau_y \in dx)/dx$ the stable(α) density as in (36).

Proof. This follows from the switching identity (17) and the definition of the T-partition. □

While the distributions of the cut times T_k and interval lengths $(T_k - T_{k-1})$ in principle determined Proposition 20, formulae for these distributions are more easily obtained as follows. For $0 < u < 1$, let

$$\tau_u^{\mathrm{br}} := \inf\{t : L_t^0 / L_1^0 = u\}$$

where $(L_t^0, 0 \leq t \leq 1)$ is the local time process at 0 of B^{br}. Then by use of the switching identity (17) we can write down for $0 < x < 1$, $0 < \ell < \infty$,

$$P(\tau_u^{\mathrm{br}} \in dx \,|\, L_1^0 = \ell)/dx = \frac{f_{u\ell}(x) f_{\bar{u}\ell}(\bar{x})}{f_\ell(1)}. \tag{63}$$

Integrating out with respect to the distribution (39) of L_1^0 gives the density

$$P(\tau_u^{\mathrm{br}} \in dx)/dx = c,(\alpha) \int_0^\infty f_{u\ell}(x) f_{\bar{u}\ell}(\bar{x}) d\ell. \tag{64}$$

which can be simplified using Lévy's formula (40) in the Brownian case to give for $\alpha = \frac{1}{2}$

$$P(\tau_u^{\mathrm{br}} \in dx)/dx = \frac{u\,\bar{u}}{2(\bar{x}u^2 + x\bar{u}^2)^{3/2}} \quad (0 < u, x < 1). \tag{65}$$

In particular, for $u = \frac{1}{2}$ we recover the the result of [9, Theorem 3.2] that $\tau^{\rm br}_{1/2}$ has uniform distribution on $[0,1]$ in the Brownian case.

According to Definition 2, $T_k := \tau^{\rm br}_{\hat{V}_k}$, for $\hat{V}_k$ independent of $B^{\rm br}$ with

$$1 - \hat{V}_k \stackrel{d}{=} \hat{V}_k - \hat{V}_{k-1} \stackrel{d}{=} \Pi_k$$

for Π_k is a product of k independent uniform$(0,1)$ variables, with

$$\frac{P(\Pi_k \in du)}{du} = \frac{(-\log u)^{k-1}}{(k-1)!} \text{ and } \sum_{k=1}^{\infty} \frac{P(\Pi_k \in du)}{du} = \frac{1}{u} \tag{66}$$

because $\log \Pi_k$ is the k'th point of a rate 1 Poisson process on $[0,\infty)$. Since the process $(\tau^{\rm br}_u, 0 \le u \le 1)$ has exchangeable increments, we find that $1 - T_k$ and the length of the kth T-interval have the common distribution

$$P(1 - T_k \in dx) = P(\lambda_{I^T_k} \in dx) = \int_0^1 P(\tau^{\rm br}_u \in dx) P(\Pi_k \in du). \tag{67}$$

In particular, in the Brownian case $\alpha = \frac{1}{2}$, (67) and (65) yield the curious formula

$$\frac{P(T_1 \in dx)}{dx} = \frac{h(x) + h(\bar{x})}{2} \text{ with } h(x) := \frac{1}{\sqrt{x}} + \log\left(\frac{1}{\sqrt{x}} - 1\right). \tag{68}$$

Corollary 21 *The point process of lengths of T-intervals has mean density*

$$\sum_{k=1}^{\infty} P(\lambda_{I^T_k} \in dx) = \alpha x^{-1}(1-x)^{\alpha-1} dx = \int_0^1 P(\tau^{\rm br}_u \in dx) u^{-1} du \tag{69}$$

for $x \in (0,1)$.

Proof. The first equality is read from part (i) of Proposition 11 and the well known formula for the mean density of points of a Poisson-Dirichlet(α) distributed sequence [32, (6)], which can be read from Lemma 15. The second equality is then read from (67) and (66). □

For general α, the second equality in (69) does not seem very obvious from (63) and (64). However, it can be checked for $\alpha = \frac{1}{2}$ using (65), and it can also be verified by a very general argument, which we indicate in Section 6.3.

Path decompositions of B^{br} at the times T_k are more complicated than the corresponding decompositions for the times D_{V_j} expressed by Lemma 8. For the T-partition, the pieces are not pure B-bridges. Rather, when normalized they have density factors involving their local times at 0. Compare with similar constructions in [11, 13, 25, 33].

By the Poisson analysis of the previous section, conditionally given $(T_1, L^0_{T_1}, L^0_1)$ the pieces of B^{br} before and after time T_1 are independent B-bridges with prescribed lengths and local times at 0. The appearance of $h+k$ in formula (a) below shows that the right side does not factor into a function of (x,h) and a function of (x,k). So even in the Brownian case, $L^0_{T_1}$ and $L^0_1 - L^0_{T_1}$ are not conditionally independent given T_1, and hence the same can be said of the fragments of B^{br} before and after time T_1.

Proposition 22 *In the Browian case with $\alpha = \frac{1}{2}$, $c = \sqrt{2}$,*

$$\frac{P(T_1 \in dx, L^0_{T_1} \in dh, L^0_1 - L^0_{T_1} \in dk)}{dx\,dh\,dk} = \frac{h\,k}{\sqrt{2\pi}} \frac{(x\bar{x})^{-\frac{3}{2}}}{(h+k)} \exp\left(-\frac{h^2}{2x} - \frac{k^2}{2\bar{x}}\right)$$

while for

$$X := \frac{L^0_{T_1}}{\sqrt{T_1}} = L^0_1(B^{\mathrm{br}}_*[0,T_1]) \text{ and } Y := \frac{L^0_1 - L^0_{T_1}}{\sqrt{1-T_1}} = L^0_1(B^{\mathrm{br}}_*[T_1,1]).$$

there is the joint density

$$\frac{P\left(X \in da, Y \in db\right)}{da\,db} = \frac{a\,b}{\sqrt{2\pi}} I(a,b) \exp\left(-\frac{a^2}{2} - \frac{b^2}{2}\right) \tag{70}$$

where

$$I(a,b) := \int_0^1 \frac{(x\bar{x})^{-1/2}}{a\sqrt{x} + b\sqrt{\bar{x}}}\,dx = \frac{1}{r}\log\left[\frac{(r+a)(r+b)}{(r-a)(r-b)}\right]$$

for $r := \sqrt{a^2+b^2}$.

Proof. The first formula is an instance of Proposition 20 which we now check. With notation as in (63),

$$P(T_1 \in dx \mid L^0_{T_1} = h, L^0_1 - L^0_{T_1} = k)/dx = f(x \mid u, \ell)$$

for $h = u\ell$ and $k = \bar{u}\ell$. We also know, by definition of T_1, that

$$P(L^0_{T_1} \in dh, L^0_1 - L^0_{T_1} \in dk) = dh\,dk\,e^{-\frac{1}{2}\ell^2}$$

where $\ell = h + k$ and an ℓ^{-1} has canceled the factor of ℓ in the density (41) of L_1^0. Combining these formulae gives the trivariate density of $(T_1, L_{T_1}^0, L_1^0 - L_{T_1}^0)$, which rescales to give

$$\frac{P\left(T_1 \in dx, X \in da, Y \in db\right)}{dx\, da\, db} = \frac{a\, b}{\sqrt{2\pi}} \frac{(x\bar{x})^{-\frac{1}{2}}}{(a\sqrt{x} + b\sqrt{\bar{x}})} \exp\left(-\frac{a^2}{2} - \frac{b^2}{2}\right).$$

and (70) follows by integrating out x. □

6 Complements

6.1 Mappings conditioned to have a single basin

In the Brownian case, a variation of the transformation from X to Y in Lemma 13, which further swaps the exchangeable pair of fragments $X[0, G_U]$ and $X[D_U, 1]$, is the continuous analog of the transformation, mentioned in Fact (2.2)(e) from the stretch of the cycles-first mapping walk for a given basin to the stretch of the basins-first walk for the same basin. As pointed out in the last section of [2], if the uniform mapping of $[n]$ is conditioned to have only one cycle, the scaled basins-first walk converges in distribution to the process $2|B_*^{\rm br}[0, D_U(B^{\rm br})]|$. The above argument yields:

Corollary 23 *For a uniform mapping of $[n]$ conditioned to have only one cycle, the scaled cycles-first walk converges in distribution to $2|B_*[0, \tau_1]|$ where $B_*[0, \tau_1]$ is the Brownian pseudo-bridge.*

The distributions of several basic functionals of pseudo-bridge $B_*[0, \tau_1]$ are known. In particular, the occupation density of the reflected process is governed by the same stochastic differential equation governing the occupation density process of a reflecting Brownian bridge or Brownian excursion [28]. According to Knight [19] (see also [33] and papers cited there), the law of the maximum of the reflected pseudo-bridge is identical to that of $1/(2\sqrt{H_1(R_3)})$ where $H_1(R_3)$ is the hitting time of 1 by the three-dimensional Bessel process, with transform $E(\exp(-\frac{1}{2}\theta^2 H_1(R_3))) = \theta / \sinh \theta$ for real θ. Thus we deduce:

Corollary 24 *For a uniform mapping of $[n]$ conditioned to have only one cycle, the asymptotic distribution of the maximum height of any tree above the cycle, normalized by $\sqrt{n}$, is the distribution of $1/\sqrt{H_1(R_3)}$.*

See also [12] for a survey of closely related distributions and their applications.

6.2 Exchangeable interval partitions

Suppose that (I_j^{ex}) is an exchangeable interval partition of $[0,1]$. That is (assuming for simplicity that the lengths $\lambda_{I_j^{\mathrm{ex}}}$ are almost surely all distinct), for each $n = 2, 3, \ldots$ such that $\lambda_{I_{(n)}^{\mathrm{ex}}} > 0$, where $(I_{(j)}^{\mathrm{ex}})$ is the associated length-ranked interval partition, conditionally given $\lambda_{I_{(n)}^{\mathrm{ex}}} > 0$ the ordering of the longest n sub-intervals $I_{(j)}^{\mathrm{ex}}, 1 \leq j \leq n$ is equally likely to be any one of the $n!$ possible orders, independently of the lengths of these n intervals. Call (I_j^{ex}) *infinite* if $P(\lambda_{I_{(n)}^{\mathrm{ex}}} > 0) = 1$ for all n. As shown by Kallenberg [16], for an infinite exchangeable interval partition (I_j^{ex}), for each $u \in [0,1]$ the fraction of the longest n intervals that lie to the left of u has an almost sure limit $\bar{L}_u^0$ as $n \to \infty$. The process $(\bar{L}_u^0, 0 \leq u \leq 1)$ is a continuous increasing process, the *normalized local time process of* (I_j^{ex}). It is easily shown that for B^{br} as in previous sections, and more generally for B^{br} the standard bridge of any nice recurrent Markov process, constructed as in [13], the interval partition (I_j^{ex}) defined by the excursions of B^{br} away from 0 is an infinite exchangeable interval partition of $[0,1]$, whose normalized local time process is $\bar{L}_u^0 = L_u^0/L_1^0, 0 \leq u \leq 1$ for any of the usual Markovian definitions of a bridge local time process $L_u^0 := L_u^0(B^{\mathrm{br}})$. In particular, this remark applies to a self-similar recurrent process B as considered in previous sections.

Theorem 25 *The assertions of Corollary 3 remain valid for the D- and T-partitions defined by Definitions 1 and 2 for any infinite exchangeable interval partition (I_j^{ex}) instead of the excursion intervals of a standard Brownian bridge B^{br}, with the complement of $\cup_j I_j^{\mathrm{ex}}$ in $[0,1]$ instead of the zero set of B^{br}, and the normalized local time process $(\bar{L}_u^0, 0 \leq u \leq 1)$ of (I_j^{ex}) instead of $(L_u^0/L_1^0, 0 \leq u \leq 1)$. Moreover, the sequence of normalized local times $(\bar{L}_{I_j}^0)$ has the same $GEM(1)$ distribution for $I_j = I_j^D$ as for $I_j = I_j^T$.*

Theorem 25 can be derived from a certain combinatorial analog, stated and proved as Lemma 26 below. Let us briefly outline the method of derivation, without details. Consider an infinite exchangeable interval partition (I_j). Take k independent uniform $(0,1)$ sample points, assign "weight" $1/k$ to each, and let $(I_j^{(k)})$ be the intervals containing at least one sample point.

Each interval $I_j^{(k)}$ is thereby assigned weight $1/k\times$ (number of sample points in interval). For fixed k we can apply Lemma 26, interpreting "length" as "weight", and conditionally on the number of intervals in the partition. The conclusion of Lemma 26 is a variant of the desired Corollary 3 for (I_j), in which "position $x \in (0,1)$" of interval endpoint is replaced by "$1/k\times$ (number of sample points in $(0,x)$)", and in which "normalized local time at $u \in (0,1)$" is replaced by "relative number of sampled intervals in $(0,u)$". One can now argue that as $k \to \infty$ we have a.s. convergence of these variant quantities to the original quantities in Theorem 25.

Lemma 26 *Let $(I_i^{\mathrm{ex}})_{1\le i\le n}$ be an exchangeable interval partition of $[0,1]$ into n subintervals of strictly positive length. Define D_{V_j} as in Definition 1 for $1 \le j \le J_n^D$, where J_n is the first j such that $D_{V_j} = 1$, to create a D-partition $(I_j^D)_{1\le j\le J_n^D}$ of $[0,1]$, and define a T-partition $(I_j^T)_{1\le j\le J_n^T}$ of $[0,1]$ similarly using cut points $T_j, 1 \le j \le J_n^T$ determined as follows: given that the random set of endpoints of $(I_j^{\mathrm{ex}})_{1\le j\le n}$ is $\mathcal{U} := \{u_j\}_{0\le j\le n}$ with $0 = u_0 < u_1 < \cdots < u_n = 1$, let T_1 have uniform distribution on $\mathcal{U}\cap(0,1]$, and given also $T_1 = t_1 < 1$ let T_2 have uniform distribution on $\mathcal{U}\cap(t_1,1]$, and so on, until $T_{J_n^T} = 1$. For I_j an interval of either of the D- or T-partitions so defined, let N_{I_j} denote the number of intervals of $(I_i^{\mathrm{ex}})_{1\le i\le n}$ which are contained in I_j, so $1 \le N_{I_j} \le n$. Then the assertions of Corollary 3 remain valid provided that N_{I_j} is substituted everywhere for $L^0_{I_j}$.*

Proof. We will check that part (i) of Corollary 3 holds in this setup, along with (72). The remaining claims are straightforward and left to the reader. By conditioning on the ranked lengths $\lambda_{I^{\mathrm{ex}}_{(j)}}$ of the intervals $(I_i^{\mathrm{ex}})_{1\le i\le n}$, it suffices to consider the case when these ranked lengths are distinct constants. Let Π_n^D denote the random partition of $[n]$ defined by the random equivalence relation $i \sim j$ iff $I^{\mathrm{ex}}_{(i)}$ and $I^{\mathrm{ex}}_{(j)}$ are part of the same component interval of the D-partition, and define Π_n^T similarly in terms of the T-partition. Since each unordered collection of lengths and sub-interval counts is a function of the corresponding partition, it suffices to show that $\Pi_n^D \stackrel{d}{=} \Pi_n^T$. Due to the well known connection between the discrete stick-breaking scheme used to define the T-partition and the cycle structure of random permutations, which was recalled in Section 2.2 (d), we can write down the distribution of Π_n^T without calculation: for each unordered partition of $[n]$ into k non-empty subsets

$\{A_1, \ldots, A_k\}$,

$$P(\Pi_n^T = \{A_1, \ldots, A_k\}) = \frac{1}{n!} \prod_{j=1}^{k} (|A_j| - 1)! \tag{71}$$

where $|A_i|$ is the number of elements of A_i. On the other hand, for the D-partition, for each ordered partition $(A_1, \ldots, A_k)$ and each choice of $a_j \in A_j$,$1 \leq j \leq k$, with $\lambda(a)$ the length of $I_{(a)}^{\text{ex}}$ and $\lambda(A) := \sum_{a \in A} \lambda(a)$, we can write down the probability

$$P(I_j^D = \cup_{a \in A_j} I_{(a)}^{\text{ex}} \text{ and } I_{(a_j)}^{\text{ex}} \text{ has right end } D_{V_j}) = \frac{1}{n!} \prod_{j=1}^{k} (|A_j|-1)! \frac{\lambda(a_j)}{\sum_{i=j}^{k} \lambda(A_i)}$$

where the factors of $(|A_j| - 1)!$ come from the different possible orderings of all but the last $I_{(i)}^{\text{ex}}$ to form I_j^D. If we now sum over all choices of $a_j \in A_j$, for each $1 \leq j \leq k$, we find that $\lambda(a_j)$ is simply replaced by $\lambda(A_j)$. If we then replace $(A_1, \ldots, A_k)$ by $(A_{\sigma(1)}, \ldots, A_{(\sigma(k)})$ and sum over all permutations σ of $[k]$, to consider all sequences of sets consistent with a given unordered partition $\{A_1, \ldots, A_k\}$, we get precisely (71) for Π_n^D instead of Π_n^T, due to the identity

$$\sum_{\sigma} \prod_{j=1}^{k} \frac{\lambda(A_{\sigma(j)})}{\sum_{i=j}^{k} \lambda(A_{\sigma(i)})} = 1.$$

This is obvious, because the product is the probability of picking the sequence of sets $(A_{\sigma(j)}, 1 \leq j \leq k)$ in a process of $\lambda(A_i)$-biased sampling of blocks of the partition $\{A_1, \ldots, A_k\}$. $\square$

We note the consequence of the previous proof that the number of components J_n^D of the D-partition and the number of components J_n^T of the T-partition have the same distribution, which is the same for every exchangeable interval partition $(I_i^{\text{ex}})_{1 \leq i \leq n}$ of $[0, 1]$ into n subintervals of strictly positive length:

$$J_n^D \stackrel{d}{=} J_n^T \stackrel{d}{=} K_n \stackrel{d}{=} \sum_{i=1}^{n} 1_{C_i} \tag{72}$$

where K_n is the number of cycles of a uniformly distributed random permutation of $[n]$, and the C_i are independent events with $P(C_i) = 1/i$. The second two of these equalities in distribution are well known and easily explained without calculation [26]. But the first is quite surprising, and we do not see how to explain it any more simply than by the previous proof.

6.3 Intensity measures

In this section we check Corollary 21 by showing it can be generalized and proved as follows:

Corollary 27 *In the setting of Theorem 25, the common intensity measure of the the point process of lengths of T-intervals and the the point process of lengths of D-intervals is*

$$\sum_{k=1}^{\infty} P(\lambda_{I_k^T} \in dx) = \sum_{k=1}^{\infty} P(\lambda_{I_k^D} \in dx) = \frac{P(D_{V_1} \in dx)}{x} = \int_0^1 \frac{P(\bar{\tau}_u \in dx)}{u} du \tag{73}$$

where $\bar{\tau}_u := \inf\{t : \bar{L}_t^0 > u\}$ is the inverse of the normalized local time process of the exchangeable interval partition.

Proof. The equality of the first three measures displayed in (73) is read from the conclusion of Theorem 25, using the fact that the D-partition is in length-biased order. The equality of the first and fourth measures follows from the definition of the T_k, the exchangeable increments of $(\bar{\tau}_u, 0 \le u \le 1)$, and (66), just as in the proof of (69). □

As a check on Theorem 25, let us verify the equality of the second and fourth measures in (73) in the following special case, which includes the setting of Corollary 21.

Let $(\tau_\ell, \ell \ge 0)$ be the inverse local time process of B at 0, for B as in Lemma 18 not necessarily self-similar. Note that we must explicitly assume $(\tau_\ell, \ell \ge 0)$ is drift free for the conclusion of part (iii) of that Lemma to be true. We assume that now. Assume that the Lévy measure of $(\tau_\ell, \ell \ge 0)$ has density $\rho(x)$. Let (I_j^{ex}) be the exchangeable partition of $[0,1]$ generated by the excursion intervals of B conditional on $B_1 = 0$ and $L_1(B) = \ell$ for some fixed $\ell > 0$, or equivalently by the jumps of $(\tau_s, 0 \le s \le \ell)$ given $\tau_\ell = 1$. Then, formula (63) generalizes easily to show that the fourth measure in (73) has density at x

$$\int_0^1 u^{-1} du \frac{f_{u\ell}(x) f_{\bar{u}\ell}(\bar{x})}{f_\ell(1)} \tag{74}$$

for $f_\ell(x)$ as in (36). On the other hand, abbreviating $D := D_{V_1}$ and $G := G_{V_1}$ so $[G, D]$ is the interval I_j^{ex} which covers the independent uniform time V_1, we know from (75) that for $0 < w < 1$

$$P(1-(D-G) \in dw) = \frac{\ell\rho(1-w)(1-w)f_\ell(w)dw}{f_\ell(1)}.$$

Also, it is easily seen that conditionally given $1-(D-G) = w$, the normalized local time $\bar{L}_G^0$ is uniform on $(0,1)$ and independent of the pair $(G, 1-D)$, which is distributed like $(\tau_{u\ell}, \tau_{\bar{u}\ell})$ conditioned on $\tau_\ell = w$. Together with the previous formula for $w = y+1-x$, this gives the trivariate density

$$\frac{P(\bar{L}_G^0 \in du, G \in dy, D \in dx)}{du\,dy\,dx} = \frac{\ell\rho(x-y)(x-y)f_{u\ell}(y)f_{\bar{u}\ell}(\bar{x})}{f_\ell(1)} (0 < y < x < 1)$$

Now (75) implies that

$$\int_0^y f_{u\ell}(y)\rho(x-y)(x-y)dy = \frac{xf_{u\ell}(x)}{u\ell}$$

so we deduce that

$$\frac{P(\bar{L}_G^0 \in du, D \in dx)}{du\,dx} = \frac{xf_{u\ell}(y)f_{\bar{u}\ell}(\bar{x})}{uf_\ell(1)}$$

and hence that the density displayed in (74) is indeed $x^{-1}P(D \in dx)/dx$.

6.4 Two orderings of a bivariate Poisson process

According to Proposition 14, for each $\alpha \in (0,1)$ the Poisson point process with intensity measure $\nu(dx, dy) = \rho(x,y)dxdy$ displayed in (33) has the following paradoxical property:

- (a) If the points (X_j, Y_j) are put in X-biased order, then the Y_j are in Y-biased order, whereas
- (b) if the points (X_j, Y_j) are put in Y-biased order, then the X_j are not in X-biased random order; even the distribution of X_1 is wrong.

We first see this for $\alpha = \frac{1}{2}$ by passage to the limit of elementary combinatorial properties of uniform random mappings. We then see it for general

α from the bridge representations of Proposition 14. Other point processes of lengths and local times with these properties can be constructed from an exchangeable interval partition, as shown by Theorem 25 in the previous section and Lemma 15. This argument, shows that (a) holds for the bivariate Poisson process with intensity (45) featured in Lemma 18, for any drift free subordinator $(\tau_y, y \geq 0)$ with $E(e^{-\xi\tau_y}) = e^{-\psi(\xi)y}$. Then the Y_j normalized by their sum have $GEM(1)$ distribution, both for an X-biased and for a Y-biased ordering. We offer here a slightly different explanation of (a) in this case. That is, given some joint density $\rho(x,y)$, we indicate conditions on ρ which are necessary and sufficient for (a) to hold for the bivariate Poisson process with intensity ρ, and then check that these conditions are in fact satisfied in the case (45).

Let (X_j, Y_j) be the points of a Poisson process on $\mathbb{R}^2_{>0}$ with intensity $\rho(x,y)dxdy$, in X-biased order. Let $\Sigma_X := \sum_j X_j$ and $\Sigma_Y := \sum_j Y_j$. Let

$$f_X(x) := P(\Sigma_X \in dx)/dx; \qquad f_Y(y) := P(\Sigma_Y \in dy)/dy$$

$$\rho_X(x) := \int_0^\infty \rho(x,y)dy; \qquad \rho_Y(y) := \int_0^\infty \rho(x,y)dx.$$

By a basic Palm calculation, as in [25]

$$P(X_1 \in dx, \Sigma_X - X_1 \in dw) = \rho_X(x)dx\, f_X(w)dw\, \frac{x}{x+w} \tag{75}$$

and similarly, with

$$f_{X,Y}(x,y) := P(\Sigma_X \in dx, \Sigma_Y \in dy)/(dxdy)$$

$$P(X_1 \in dx, Y_1 \in dy, \Sigma_X - X_1 \in dw, \Sigma_Y - Y_1 \in dv) \tag{76}$$
$$= \rho(x,y)\, dx\, dy\, f_{X,Y}(w,v)\, dw\, dv\, \frac{x}{x+w}.$$

Now, a *necessary condition* for the Y_j to be in Y-biased order is that Y_1 should have the same joint distribution with Σ_Y as if Y_1 were a size-biased pick from the Y_i, that is like (75)

$$P(Y_1 \in dy, \Sigma_Y - Y_1 \in dv) = \rho_Y(y)dy\, f_Y(v)dv\, \frac{y}{y+v}. \tag{77}$$

Thus a necessary condition on $\rho(x,y)$ for (a) to hold is that for all $y, v \geq 0$

$$\int_0^\infty dx \int_0^\infty dw \rho(x,y) f_{X,Y}(w,v) \frac{x}{x+w} = \rho_Y(y) f_Y(v) \frac{y}{y+v}. \tag{78}$$

Moreover, by keeping track of the first k of the (X_j, Y_j) jointly with Σ_X and Σ_Y it is clear that we can write down a multivariate version of (78) whose truth for all k would be necessary and sufficient for (a).

In the special case (45), with $f_y(x) := P(\tau_y \in dx)/dx$, the subordination argument gives

$$\rho(x, y) = y^{-1} f_y(x) e^{-\xi x}.$$

Since the Y-marginal is exponential with rate $\psi(\xi)$,

$$f_Y(y) = \psi(\xi) y \rho_Y(y) = \psi(\xi) e^{-\psi(\xi) y}$$

and hence by generalization of (43), using $(\Sigma_X, \Sigma_Y) \stackrel{d}{=} (G, L)$,

$$f_{X,Y}(x, y) = \psi(\xi) f_y(x) e^{-\xi x} = \psi(\xi) y \rho(x, y).$$

If these expressions are substituted in (78), and we use the definition of $\psi(\xi)$ on the right side, we find that (78) reduces to the identity

$$E\left[\frac{\tau_y}{\tau_y + \tau_v e^{-\xi(\tau_y + \tau_v)}}\right] = \frac{y}{y+v} E\left[e^{-\xi(\tau_y + \tau_v)}\right].$$

But this is true by virtue of

$$E\left[\frac{\tau_y}{\tau_y + \tau_v} \,\middle|\, \tau_y + \tau_v\right] = \frac{y}{y+v}$$

which holds by exchangeability of increments of $(\tau_\ell, \ell \geq 0)$. Moreover, the multivariate form of (78) mentioned above is easily checked the same way.

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